Analysis I – Week 10 Review

Abstract

These questions are intended to help you to review this week's contents; please take the time to do that, come up with examples and play around with the concepts. If you encounter problems, take actions soon.

1. We started the week by making the marvellous discovery that

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$$

converges and I told you that

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} = \ln(2). \tag{1}$$

For the moment, without having all the tools at out disposal so far, you can convince yourselves as follows. Develop $\ln(1+x)$ into its McLaurin series and take $x = 1$. Why does the series converge? Apply Abel's Theorem to reach the conclusion that (1) is indeed true. (Do this only if you have understood the rest of this week's material.)

2. We then discovered that it seems that we can rearrange the summation of $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$ in such a way that we get a different sum. This is puzzeling.

3. To make sure that we stand on solid ground for everything we have done so far, we assured ourself that finite sums do not change their sum upon rearrangement of the summands and that infinite sums of non-negative summands can also be rearranged without changing their sum.

4. After that we were interested to understand the phenomenon better. The first step was to get another convergence test for a special class of series. Let (a_n) be a decreasing null-sequence, then the series

$$\sum_{k=1}^{+\infty} (-1)^{k+1} a_k$$

converges.

5. This gives us that $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^2}$ is convergent. To further our insights into the matter, we gave a different proof of this fact by introducing two new series $\sum u_n$ and $\sum v_n$ for a given series $\sum a_n$, where we defined

$$u_n := \frac{1}{2}(|a_n| + a_n), \quad v_n := \frac{1}{2}(|a_n| - a_n).$$

We obtain $0 \leq u_n \leq |a_n|$ and $0 \leq v_n \leq |a_n|$ which implies that $\sum u_n$ and $\sum v_n$ converge since $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges.

6. We realised that the convergence of $\sum |a_n|$ has some significance as

$$\sum_{k=1}^{+\infty} \left| \frac{(-1)^{k+1}}{k} \right|$$

diverges and

$$\sum_{k=1}^{+\infty} \left| \frac{(-1)^{k+1}}{k^2} \right|$$

converges. However, both

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} \text{ and } \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^2}$$

converge. We introduced the notion of absolute convergence and clarified the relationship to convergence.

7. As a final result for this week, we proved that absolutely convergent series can be rearranged arbitrarily without changing their sum, i.e. the summands of

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^2}$$

con not be rearrange to change its sum though it has infinitely many negative elements as $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$. So, we found that absolute convergence is the right notion of convergence if we want to be sure that we can rearrange the summands of series without changing sum. Make sure you understand that a convergent series with non-negative summands is absolutely convergent.

8. This is an extra piece: It is known that

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{2}$$

With that, one can compute that

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}. \tag{3}$$

Can you, assuming (2) to be true, show (3).