Analysis I – Week 11 Review

Abstract

Before you go on Holiday, make sure that you have everything you need for your review. Read the given advice carefully and make reasonable adjustments to your learning habits. Remember that everything in Analysis 1 is basis for our discussions in Analysis 2.
Series are usually hard for most people who first encounter them. Be prepared that some concepts will take you some hours of hard work pondering to digest.

1. Do you understand that for series $\sum_{n=1}^{+\infty} a_n$ with only positive summands, i.e. $a_n \geq 0$ for all $n \in \mathbb{N}$.

2. To understand the Riemann Rearrangement Theorem better, watch the mathologer video Riemann's paradox $\pi = \infty - \infty$.

3. We discussed the fantastic Riemann Rearrangement Theorem which shows just how weird conditionally convergent series are. To understand the theorem better and understand why it is even possible, you first need to understand what we have done before. After that, discuss the following points: Let $\sum a_n$ be conditionally convergent.

   (a) Clarify that there must be infinitely many $a_n < 0$.

   (b) Investigate the (absolute) convergence of the series $\sum u_n$ and $\sum v_n$, where

   $$u_n := \frac{1}{2}(|a_n| + a_n), \quad v_n := \frac{1}{2}(|a_n| - a_n).$$

   See also the comment at the Riemann rearrangement theorem in the completed notes.

   (c) How can that be used to prove that for any $R \in \mathbb{R}$, there is a rearrangement $\sum b_n$ of $\sum a_n$ such that $\sum b_n = R$. You want to be able to sum a part of the series to get a partial sum larger than R then you want to add some negative elements to get smaller than R and so forth. Can you make this a bit more precise?

4. Here we have another example of a sub-space of a vector space. We proved (Where?) that the set of convergent series is a vector space. One can show that the set of absolute convergent series is a sub-space. See your Linear Algebra lecture notes.

5. The final object of the module is a *power series centred at* x_0 (or at a in many places of the notes.) which is an expression of the type

$$\sum_{n=0}^{+\infty} c_n (x - x_0)^n$$

where the c_n are called coefficients and the powers $(x - x_0)^n$ give the power series their name.

6. We can associate a function $f : D_R(x_0) \to \mathbb{R}$ to a power series by

$$\begin{cases} f : D_R(x_0) \to \mathbb{R} \\ x \mapsto f(x) = \sum_{k=0}^{+\infty} c_k (x - x_0)^k \end{cases} .$$

We set $D_R := \{x \in \mathbb{R} : |x - x_0| < R\}$, where R is the radius of convergence.

7. Sometimes people say that power series are polynomials of infinite degree. However, as I have discussed in class, this interpretation is a bit of a stretch. What one can say is that polynomials are power series where only finitely many of the c_n are different from 0. (One can show that polynomials of any degree form a subspace of the vector space of convergent power series.)

8. We extended the ratio test for series $\sum_{n=1}^{+\infty} a_n$ to say that if

$$\lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| \leq q < 1,$$

then $\sum_{n=1}^{+\infty} a_n$ is convergent (in fact absolutely convergent). See also Review Sheet 9. This helped us to investigate

9. We made some connections between power series and material that you have seen in Mathematical Methods 1. You should consult your notes on Taylor- and MacLaurin series and refresh your memory.

10. Finally, remember that mathematics does not consist of isolated islands of facts and that you should always think about the big picture and how the different modules you have are interconnected. You will not get everything spoon fed. To further your general education you might want to have an look at the following articles:

- M. Atiyah The unity of Mathematics (Some of the mathematics discussed there is far beyond what you know so far. The points can be understood nevertheless and maybe ven better as they are connected with quite some elementary mathematics.)
- S. Lang The Beauty of Doing Mathematics.
- W. Thurston On proof and progress in mathematics
- Mathematical Beauty Wikipedia article.