Analysis I – Week 2 Review

1. Picking up the discussion from last week, people are sometimes troubled by mathematicians making them study axioms. They think: 'But these are obvious'. But philosophically they are less obvious than you might think.

   (a) It is not as though anyone could check that every real number really has a multiplicative inverse. This means that if we want to prove that something is true for every real number, we have to assume that things like this are true, i.e. when we say x is a real number we mean it belong to a set of objects that satisfy the axioms we discussed.

   (b) It took mathematicians about 2500 years[1] to figure this out. If you do not yet get the necessity, you will at some point in your studies. However, at the end of this semester you can for certain see that quite astonishing results flow from those simple rules. Give it some time.

2. **Mathematical notation**. I have seen that some students write $a \times b$ for $a \cdot b = ab$. If I see such thing in the tests or exams, you will get zero marks. The $\times$ sign has reserved meaning and the multiplication is denoted by a centred dot as in $a \cdot b$. For $a.b$, you will get zero marks as well since that is notation that we use to write things as 2.14. Please adopt all notation as soon as it is introduced! Concerning notation, it does not mater what you did in school. Now you are on the way to become a mathematician and you should adopt the standard way of writing. You find information on set notation in Chapter 0 of the Lecture Notes.

3. We have now completed our discussion of the axioms of the real numbers. You can convince yourself that $\mathbb{Q}$ or $\mathbb{Q}[\sqrt{2}]$, which you have seen in Linear Algebra, satisfies all axioms of addition and multiplication as well as the distributive law; for short, we refer to them as the *field axioms*. See also the schematic below. Another interesting one is $\mathbb{F}_2 = (\{0, 1\}, +, \cdot)$, where the operations are defined

[1] Counted from the advent of the Greek mathematics. However, branches of algebra and geometry are as old as five and six thousand years and were studied in the early periods of Mesopotamia and Babylon.

by

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\,, \quad \text{and} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}\,.$$

All we[2] have to do is to see that it satisfies the axioms, then we now that everything we concluded from the axioms will be true. Not all algebraic structures you have seen so far satisfy all axioms. For example, the set $M_n(\mathbb{R})$ of all $n \times n$ matrices with real entries with the usual operations, satisfies all laws for addition, some for multiplication and two distributive laws. You will see many more during your studies in later semesters.

<table>
<tr><td colspan="2">Field axioms: There are two operations (addition ($+$), multiplication ($\cdot$)) which satisfy:</td><td rowspan="3">Field axioms</td><td rowspan="6">Ordered field axioms</td><td rowspan="7">Complete ordered field axioms</td></tr>
<tr><td>Axioms of addition
Associativity law
Commutativity law
Existence of the zero
Existence of the negative</td><td>Axioms of multiplication
Associativity law
Commutativity law
Existence of the one ($\neq 0$)
Existence of the inverse ($\neq 0$)</td></tr>
<tr><td colspan="2">Distributive law</td></tr>
<tr><td colspan="3">Order axioms: there are some elements that are denoted as positive ($x > 0$) such that the following axioms are satisfied</td></tr>
<tr><td colspan="3">Trichotomy: For all elements x and y in $\mathbb{R}$ exactly one of the following is true:
$x < y, \quad x = y, \quad y < x$</td></tr>
<tr><td colspan="3">$x < y$ and $y < z \Rightarrow x < z$ (Transitivity)
$x < y \quad \Rightarrow \quad x + z < y + z$ for all $z \in \mathbb{R}$ (Compatible with $+$)
$x < y$ and $z > 0 \Rightarrow xz < yz$. (Compatible with $\cdot$)</td></tr>
<tr><td colspan="4">Completeness axiom:
Every non-empty bounded set $A \subseteq \mathbb{R}$ has a supremum.</td></tr>
</table>

Table 1: Schematic classification of the axiom of the set of the real numbers $\mathbb{R}$.

4. This weeks main point was the *completeness axiom*. We started some discussion of its consequences and how that sets apart $\mathbb{Q}$ from $\mathbb{R}$. We will continue those discussions next week.

Let us repeat what we had so far. We wanted to see that the set S, defined by

$$S := \{p \in \mathbb{Q} : p^2 < 2\} \subseteq \mathbb{Q}$$

[2]That means you. This is something for a rainy November afternoon with a cup of hot cocoa.

has no supremum in $\mathbb{Q}$. The set is clearly bounded above as $3 \in \mathbb{Q}$ is an upper bound. As a sub-set of $\mathbb{R}$, the set S has a supremum and that is $\sqrt{2}$.[3] What remains to show is that for any element p in S there exists a q in S such that $q > p$. I presented you

$$q = \frac{2p+2}{p+2}$$

as such a candidate. No worry where that came from. The right thing to worry about is the method of proof and the goal of the result. I gave you the hint, that you should continue like

$$q = \frac{2p+2+p^2-p^2}{p+2} = \frac{p(2+p)+2-p^2}{p+2} = p + \frac{2-p^2}{p+2}$$

So, what do we know about $p^2 - 2$ and p? Remember that they are in S. Can we conclude that $q > p$? Next, show that $q^2 < 2$ to see that $q \in S$.

5. In preparation of the class test let me reiterate how you should study:

   (a) Come to the lectures and take useful notes. You achieve that by listening and comprehending not by robotic copying. *There is no fear to miss something as there are completed notes.* The writing however, will help you remember and digest right away.

   (b) Revise your notes, add comments and compare to the completed notes. If you have questions, write them in a list that you should keep handy. Use it in meetings with me, your tutors, and your friends.

   (c) Try the problem sheets and actually do the exercises I give you in class. You must loose your fear of being wrong. There is no progress without discussion and in a discussion one is allowed to err. Make use of the small group tutorials.

[3] It is a valuable exercise to revisit the proof that $\sqrt{2}$ is irrational. You can find this proof in the Lecture Notes in the historical remarks in Chapter 1.