Analysis I – Week 2 Review

1. We managed the first chapter of Analysis 1 and I believe that the coming two will be a lot more interesting to some people as they are a bit less abstract. We are about to embark on our journey through the magic gardens of sequences and series and will learn marvellous things.

2. A word to the problem sheet. Some of you have worked on the sheets and still have not gotten all the problems. That is not bad but normal at the start. Over the course of time, it should get better but at the beginning you also have to learn to live with some frustration. Use Polya's algorithm and ask yourself: **Have I made use of all the tools Christian has made me aware of that I need them all the time?** For the next lectures that will be:

   (a) Our mantra of *adding an active zero* and *multiplying an active one* and

   (b) the *triangle inequality/reverse triangle inequality*.

   For example: We want to show that $|a-b| \leq |a-c|+|c-b|$ for arbitrary $a, b, c \in \mathbb{R}$. How does that look? Well, clearly that screams triangle inequality. So, we start with what we know and knowing where we want to go:

   $$|a-b| = |a-c+c-b| \leq |a-c| + |c-b|,$$

   where we used the triangle inequality $|x+y| \leq |x| + |y|$ for $x = a-c$, $y = c-b$. This is a typical use of the inequality.

3. We used the completeness axiom to prove that every real $x \geq 0$ has a real square root $c \geq 0$, i.e. number such that $c^2 = x$. I wish that you take away the following strategy of proof which is a typical application of the completeness axiom:

   (a) Define a set M in such a way that we can reasonable expect that the supremum of the set has the desired properties.

   (b) Prove that the set is bounded above (or below) and apply the completeness axiom.

(c) Prove that sup(M) has indeed the desired property.

At the end of this sheet, I will reproduce the proof in detail but this is one of the few proofs you need not to be able to reproduce. You are however required to understand and remember the strategy of proof.

4. We have introduced the notion of the absolute value (or modulus) of a real number x, denoted by $|x|$. From the definition, it immediately follows that $|x| \geq 0$. If you draw a number line, it should be clear that $|x|$ is the distance of x to 0. To prepare for the next lecture, you might want to think about how you would define the distance between to arbitrary numbers a and b. What properties would you expect the distance to have?

   (a) What is the relation of the distance from a to b and from b to a?

   (b) Should we expect some relation between the distance from a to b and from a to c and c to b? Draw number lines with c in between a and b and outside of the interval defined by a and b. What can you guess for the distances?

5. To familiarise yourself with the new notion, prove the remaining facts in Theorem 1.2. If you have no better idea, you should simply proceed by a reasonable case study, e.g. $x > 0$, $x < 0$, and $x = 0$. See also that $|x^2| = |x|^2$ follows of course from $|xy| = |x||y|$ which tells us that we should just prove the latter. Let me do that in part to give you a hint: By our discussion in Chapter 1, we have that $xy > 0$ if either $x > 0$, $y > 0$ or $x < 0$, $y < 0$. Thus, if $xy > 0$, and $x, y > 0$ we get $|xy| = xy = |x||y|$ by the definition of the absolute value. If $xy > 0$, and $x, y < 0$, we get $|xy| = xy = (-|x|)(-|y|) = |x||y|$ from the definition of the absolute value.[1] The rest of the argument, $xy < 0$, I leave to you now.

[1]Since $|x| = -x$ for negative x.

Here, I give the complete proof of the existence of non-negative real square roots of non-negative real numbers. The proof of uniqueness is complete in your notes. See also my remark in Number 3 on this sheet.

> For every $x \in \mathbb{R}$, $x \geq 0$, there exists (exactly one) $c \in \mathbb{R}$, $c \geq 0$ such that $c^2 = x$.

Proof. We define the set

$$M := \{z \in \mathbb{R} : z \geq 0,\ z^2 \leq x\}.$$

Since we have $(1+x)^2 \geq 1+x$ (remember that $x \geq 0$), we get

$$z^2 \leq x \leq 1+x \leq (1+x)^2.$$ [2]

Thus, we get that $z \leq x+1$. Thus, the set M is bounded above. By the completeness axiom, we then get that sup(M) exists. We claim that $c^2 = x$ with $c :=$ sup(M).

The complicated part of the proof is to show that. We will do that by using the **trichotomy**. We will show that $c^2 < x$ and $c^2 > x$ leads to a contradiction. Thus, we must have $c^2 = x$.

1. Let $c^2 < x$: Then, by choosing

   $$\varepsilon := \text{min}\left\{1, \frac{x-c^2}{2c+1}\right\} > 0,$$

   we get that

   $$(c+\varepsilon)^2 \leq c \tag{1}$$

   which leads to $x+\varepsilon \in M$ and $x+\varepsilon \leq x$. However, this is a contradiction since $\varepsilon > 0$.

   To see (1), we calculate $(c+\varepsilon)^2$:[3]

   $$(c+\varepsilon)^2 = c^2 + 2\varepsilon c + \varepsilon^2 \leq c^2 + \varepsilon(2c+1) \leq x,$$

[2] The question you ask is, how can I estimate z against something and what you know is that z^2 is bounded by something. So, if you can bound z^2 by another square, you get the right estimate after some pondering. As ever, know what you want.

[3] This is in fact how we calculate ε but when presenting the argument the choice of ε is only in so far important as that it does its job.

where we used $\varepsilon^2 \leq \varepsilon$.

2. Let $c^2 > x$. In this case, x and c are larger than 0. Then, if you choose $\varepsilon > 0$ by

$$\varepsilon := \text{min}\left\{\frac{x}{2}, \frac{c^2 - x}{2x}\right\} > 0,$$

we get that $x - \varepsilon \geq x - \frac{x}{2} = \frac{x}{2} > 0$ and also

$$-2\varepsilon x \geq x - c^2$$

and then

$$(c - \varepsilon)^2 = c^2 - 2\varepsilon c + \varepsilon^2 > x.$$[4]

Since $z^2 \leq x$ for all $z \in M$, we then get that $z^2 < (c - \varepsilon)^2$ for all $z \in M$. This implies $z < c - \varepsilon$ for all $z \in M$ which contradicts that c is the supremum of M.

This concludes the proof since the trichotomy[5] gives us that $c^2 = x$.

[4] Again, to find ε, you would first do the calculation and then see how you have to choose the ε.
[5] Remember, for two real numbers a and b exactly one of the following is true: $a < b$, $b < a$, or $a = b$.