Analysis I – Week 6 Review

1. This week we learned a couple of important theorems. Can you recall their correct statements?

   (a) Arithmetic properties of convergent sequences.[1]

   (b) Sandwich theorem[2]

   (c) The characterisation of null-sequences.

   (d) If for a sequence (a_n) you know that $(|a_n|) \to |a|$, what can you say about the convergence of (a_n)?

   (e) What is the connection of convergence with boundedness? Can you explain that to a fellow student with an illustration on the number line?

   (f) Is there a sequence that converges to 1 and to 5? Can you explain that with an illustration on the number line.

2. Can you prove every of the above made assertions?

3. If I told you that every bounded sequence has a convergent sub-sequence, can you come up with a strategy to prove that? What would we need to prove? I do not mean you should try (yet), only try to come up with a strategy. (Hint: What do we need to investigate? We have not too many theorems about sub-sequences.)[3]

4. Calculate the limits of the following sequences (a_n), defined by

   (a) $a_n := \dfrac{\sin(n^2+7n+3)}{n^2+1}$,

   (b) $a_n := \dfrac{n^{17}+3n^3}{\sin(n^2)+2n^{30}}$,

   (c) $a_n := \dfrac{\sin(n!)}{n!}$,

   (d) $a_n := \dfrac{3n^3}{2+\sin^2(n^2)}$,

   (e) $a_n := \dfrac{2n^2+8n+2}{5n^2+3n+2}$,

   (f) $a_n := \dfrac{\sqrt{n}+2}{\sqrt{n}+1+\sin(n^2)}$.

5. If you feel secure about everything we have done so far, you might want to try to come up with a result for $(x^{\frac{1}{n}}) = (\sqrt[n]{x})$ for a real number $x > 1$. We will discuss this later in class but in principle you know everything to prove a limit theorem. However, if you are struggling, you should work on the basics, see Point 1. (Hint: It is a tricky application of Bernoulli's inequality. Note that $x^{\frac{1}{n}} > 1$ for all $n \geq 1$. Thus, $x^{\frac{1}{n}} - 1 = y \geq -1$. Then, $x^{\frac{1}{n}} = 1 + y$. See where this is going?)

[1] Do you understand that they imply that the set of convergent sequences is a vector space?
[2] Or Squeezing Theorem or Policemen principle.
[3] A look at Problem Sheet 5 might give you a hint?!