Analysis I – Week 7 Review

Abstract

You not only need to get things into your memory, you need not to forget that it takes also training to get it out.
Do not cram your learning too much. Use the time you have now. This is the last time in your life that you can spend exclusively on learning exciting things. Do not throw it away.

1. It is now time to make a complete overview over the Chapter Sequences.

   (a) Do a schematic overview on concepts and how they are related. See the example in the completed noes for instance.

   (b) Take a blank sheet of paper and write on top something like boundedness, convergence, monotonicity, etc. pp. Then, write every definition and related theorem related to the topic that you remember. Complete with examples and the main ingredient of the proof that you remember, e.g. Bernoulli's inequality & sandwich theorem etc. pp.

2. This week, we learned a lot of standard limits which are so-called as you can use them from now on in calculations, especially in the exam. When you use result like the one on reciprocals of sequences tending to infinity, you should indicate that by something like *by a result of the lecture*. Results with names as the Sandwich Theorem & d'Alembert's ratio test should be stated with name.

3. Related to point two. Make sure that you know all of these nd that you know the main ingredients of the proofs.[1]

   (a) $\lim\limits_{n\to+\infty} n^\alpha = \begin{cases} +\infty & : \alpha > 0 \\ 1 & : \alpha = 0 \\ 0 & : \alpha < 0 \end{cases}$,

   (b) $\lim\limits_{n\to+\infty} x^n = \begin{cases} +\infty & : x > 1 \\ 1 & : x = 1 \\ 0 & : -1 < x < 1 \end{cases}$,

   (c) $\lim\limits_{n\to+\infty} \sqrt[n]{x} = 1$ for $x > 0$,

   (d) $\lim\limits_{n\to+\infty} \sqrt[n]{n} = 1$.

[1] Bernoulli's inequality, the Sandwich Theorem, and a result on reciprocals of sequences tending to infinity.

4. We also discussed how to handle sequences of the type

$$\left(\sqrt[n]{x^n + y^n}\right)_{n\in\mathbb{N}}$$

for $0 \le x < y$. (What happens if $x = y$?) The last result also gave us how to handle sequences of the type

$$\left(\sqrt[n]{C + x^n}\right)_{n\in\mathbb{N}}.$$

Work this out in greater detail, in general and in examples.

5. We proved that monotone and bounded sequences converge. With that, prove that the sequence $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 2}$ converges. We already proved that, if the sequence converges, the limit must be equal to 2.

   (a) Prove that the elements of the sequence are positive and the sequence is increasing.

   (b) Prove that the sequence is bounded.

   (c) Conclude the convergence.

6. On Wednesday, we discussed the concept of Cauchy sequences. Here, instead of controlling the distance of the elements a_n of (a_n) to a limit a, we control the distance of all possible a_l and a_k from some index on, i.e. we suppose some sort of lumping of the elements of a.

7. Do you understand the definition? Can you draw the related pictures from memory.

8. We then proved that a sequence converges (a_n), i.e.

$$\exists a \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \ : \ \forall n \ge n_0, \ |a_n - a| < \varepsilon$$

is equivalent to

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \ : \ \forall n, m \in \mathbb{N}, \, n, m \ge n_0, \ |a_n - a_m| < \varepsilon.$$

Work, with pen and paper, again through all steps of that proof.

9. The great advantage of Cauchy sequences is that we can use it to prove the convergence of a sequence, without knowing the limit. On the other hand, this can be seen as a disadvantage too. See Number 5.