Analysis I – Week 8 Review

1. This week, we introduced infinite sums which we called series. The first problem is to properly define, what that is supposed to be and what a result of an infinite sum $\sum_{k=1}^{n} a_k$could be. One would probably start by saying, a series has a sum S iff for all $\varepsilon > 0$, there exits an $n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N},\ n \geq n_0, \quad \left| \sum_{k=1}^{n} a_k - S \right| < \varepsilon.$$

We did that by introducing the notion of the partial sum

$$s_n := \sum_{k=1}^{n} a_k$$

of the series under consideration. We then defined terms as convergence and divergence in terms of the sequence (s_n). Note that the above relation is exactly $(s_n) \to S$.

2. We[1] proved then that the series

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} \tag{1}$$

is convergent. For that we first observed that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{k+1}.$$

With that, we can prove that

$$\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} \tag{2}$$

is convergent. First, we look at the partial sums

$$t_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1} \leq 1.$$

Further, we have

$$t_{n+1} - t_n = \frac{1}{(n+1)(n+2)} > 0,$$

[1]Here, I streamlined the notation a bit here. The t_n is now the partial sum of $\sum \frac{1}{n(n+1)}$. It would be best If you take a blank sheet of paper and work through this proof with this sheet and your notes writing everything out again.

i.e. $t_{n+1} > t_n$. Thus, we have that (t_n) is increasing and bounded and therefore convergent. Thus (2) converges. Now let us connect that to the original question: the partial sum of (1) is

$$\begin{aligned} s_n = \sum_{k=1}^{n} \frac{1}{k^2} &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} = 1 + \sum_{k=1}^{n-1} \frac{1}{(k+1)^2} \\ &\leq 1 + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} = 1 + t_{n-1}. \end{aligned}$$

We further note that $s_{n+1} > s_n$. Thus, (s_n) converges by the shift theorem and the sum rule for convergent sequences together with the monotone convergence theorem.[2]

3. The second example we looked at was a divergent one. We proved that

$$\sum_{k=1}^{+\infty} \frac{1}{k}$$

tends to infinity. The proof of this is on the problem sheet for this week. Work through it to understand the construction better. Before you do so you should make sure tht you do understand the hand waving argument right at the start

$$\sum_{k=1}^{+\infty} \frac{1}{k} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \frac{1}{9} + \frac{1}{10} + \ldots .$$

4. After that we considered a shift theorem for series as well as a comparison theorem. Your reading was about the very important geometric series and if you have not done it so far you should get at it. The reading contained also a ratio test for series which utilised ideas from the ratio test for sequences.

5. As for sequences, we have asked ourselves whether we can add convergent series and answered this question positively. Which that we have proved that the set of convergent series is a vector space. For that see also you Linear Algebra notes.

6. The final result of this week's lectures was the null-sequence test which help us to identify some divergent series as for $\sum_{n=1}^{+\infty} a_n$ to be convergent, $(a_n) \to 0$ is

[2]Shorthand for *every monotone and bounded sequence is convergent.*

necessary. However, we also know that it is not sufficient as $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges.