Analysis I – Week 9 Review

Abstract

These questions are intended to help you to review this week's contents; please take the time to do that, come up with examples and play around with the concepts. If you encounter problems, take actions soon.

1. Recall that to any series $\sum_{k=1}^{+\infty} a_k$ is associated a sequences (s_n), called the sequence of partial sums, defined by

$$s_n := \sum_{k=1}^{n} a_k, \quad \text{and,}$$

if (s_n) is convergent, we say

$$\sum_{k=1}^{+\infty} a_k = \lim_{n\to+\infty} \sum_{k=1}^{n} a_k$$

2. We discussed that the set of convergent sums is a vector space, i.e. if $\sum_{k=1}^{+\infty} a_k$ and $\sum_{k=1}^{+\infty} b_k$ are convergent them, for all $\lambda, \mu \in \mathbb{R}$, the series

$$\sum_{k=1}^{+\infty} \lambda a_k + \mu b_k$$

converges and

$$\sum_{k=1}^{+\infty} (\lambda a_k + \mu b_k) = \lambda \sum_{k=1}^{+\infty} a_k + \mu \sum_{k=1}^{+\infty} b_k.$$

As in the case of sequences, we are aware that the converse is not true, i.e. if we have that

$$\sum_{k=1}^{+\infty} a_k + b_k$$

converges, we can not say anything about $\sum_{k=1}^{+\infty} a_k$ and $\sum_{k=1}^{+\infty} b_k$.

3. We had a glimpse on a possible discussion of products of series but we have also seen that this is, at the moment, beyond are abilities as it seems to need a discussion of sequences depending on more than one index.

However, I encourage you to have a look in the literature, especially concerning a special kind of product, the Cauchy product. This one is important for Power series, for instance Taylor expansions of analytic functions.

4. We have learned several ratio tests by now. The convergence part of the ratio test for series says that if $a_n > 0$ for all $n \in \mathbb{N}$, we have that

$$\sum_{k=1}^{+\infty} a_k$$

is convergent if

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l \in [0,1).$$

Can you give an argument why this test still applies even if we do not suppose that the summands are all positive, i.e. we claim that

$$\sum_{k=1}^{+\infty} a_k$$

converges if

$$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = l \in [0,1).$$

Investigate the convergence of

$$\sum_{k=1}^{+\infty} \frac{k^5}{2^k}, \quad \sum_{k=1}^{+\infty} (-1)^k \frac{k^{11}}{3^k}, \quad \sum_{k=1}^{+\infty} \frac{(-1)^k k^4 9^k}{10^k}.$$

5. We then considered how we can utilise what we know about some 'standard' series to get information on more complicated ones. The most important 'standard' series are

$$\sum_{k=1}^{+\infty} \frac{1}{k^\alpha} = \begin{cases} \text{divergent (tending to } +\infty) & : \alpha \leq 1, \\ \text{convergent} & : \alpha > 1. \end{cases}$$

We proved two comparison results for which these standard series are helpful. Can you state the results? Can you repeat the proof? Can you give a different proof? Give examples of applications. Decide, using appropriate ratio tests, whether the following series are convergent :

$$\sum_{k=1}^{+\infty} \frac{n^2+2n+3}{n^5+2n^2+3n+7}, \quad \sum_{k=1}^{+\infty} \frac{n^2+\ln(n)}{n^3+2n+3}, \quad \sum_{k=1}^{+\infty} \frac{k}{2^k}.$$

For the last one you might want to consult the last question on Problem Sheet 7. You can even find its sum. Also number 4 on this sheet applies of course.

6. Generalising from Problem Sheet 7, can you get a proof for

$$\sum_{k=1}^{+\infty} k^2 q^k = \frac{q(q+1)}{(1-q)^3}$$

   With methods from the Power Series Section of Mathematical Methods 1, you can develop a different proof as well. You can differentiate $\sum_{k=0}^{+\infty} q^k$ with respect to q twice (and do some algebra). (After one time, you get the formula on Problem Sheet 7.)

7. You should make a list of convergence tests so that you have quick access when you are looking for something to solve problems on the problem sheet.