# Uniform continuity

Two examples.

1. Let us recall the definition of a function $f : (a, b) \to \mathbb{R}$ being continuous on $(a, b)$: For all $x_0 \in (a, b)$ and all $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Important is, that we can not avoid, in general, that $\delta$ depends not only on $\varepsilon$ but on the point $x_0$ as well. An example, with the appropriate interpretations at the boundary, is $f : [0, +\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$, where we have to choose $\delta = \sqrt{x_0}\varepsilon$. Can we choose $\delta$ in a different way such that it does not depend on $x_0$? Let us first consider $f$ on $[1, +\infty)$:

$$|f(x)-f(x_0)| = \left|(\sqrt{x} - \sqrt{x_0})\frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}}\right| \leq \frac{1}{1 + \sqrt{x_0}}|x-x_0| \leq \frac{1}{2}|x-x_0| < |x-x_0|.$$

Which gives us

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

for $\delta = \varepsilon$. Now let us look at $f$ on $[0, 1]$. By the triangle inequality, we get

$$|\sqrt{x} - \sqrt{x_0}| \leq |\sqrt{x}| + |\sqrt{x_0}| = |\sqrt{x} + \sqrt{x_0}|$$

which gives

$$\begin{aligned} |\sqrt{x} - \sqrt{x_0}|^2 &= |\sqrt{x} - \sqrt{x_0}||\sqrt{x} - \sqrt{x_0}| \\ &\leq |\sqrt{x} - \sqrt{x_0}||\sqrt{x} + \sqrt{x_0}| = |x - x_0| \end{aligned}$$

which yields

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

for $\delta = \varepsilon^2$. Thus, we can choose $\delta = \varepsilon^2$ on $[0, +\infty)$.

2. Let us consider $f : (0, 1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. We compute

$$|f(x_0) - f(x)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x_0 x}\right| \leq \frac{1}{|x_0||x|}|x - x_0| = \frac{1}{x_0} \cdot \frac{1}{x} \cdot |x - x_0|.$$

Now, from $|x - x_0| < \delta$, we obtain

$$x_0 - \delta < x < x_0 + \delta \quad \Rightarrow \quad \frac{1}{x} < \frac{1}{x_0 - \delta}.$$

Assume $\delta \leq \frac{x_0}{2}$, which gives us $\frac{1}{x} < \frac{1}{x_0 - \delta} \leq \frac{2}{x_0}$. Thus, we get

$$|f(x_0) - f(x)| \leq \frac{1}{x_0 x}|x - x_0| \leq \frac{1}{x_0}\frac{2}{x_0}|x - x_0|.$$

Hence, the choice $\delta = \min\left\{\frac{x_0}{2}, \frac{x_0^2 \varepsilon}{2}\right\}$ yields

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Thus, $f$ is continuous on $(0, 1)$. However, we will see that it is not uniformly continuous. To prove that, let us assume that $f$ is uniformly continuous, i.e. for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$. Now, we set $\varepsilon = \frac{1}{2}$, i.e. we have a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \frac{1}{2}$. Thus, we can choose a number $\eta > 0$ such that $\eta < \delta$ and set $x = \eta$ and $y = \frac{\eta}{2}$. Thus,

$$|x - y| = \left|\frac{\eta}{2}\right| < \eta < \delta.$$

Now, we get

$$|f(x) - f(y)| = \left|\frac{1}{\eta} - \frac{2}{\eta}\right| = \frac{1}{\eta} > \frac{1}{2}$$

if $\eta > 0$ is small enough. Thus, we reached a contradiction and $f$ can not be uniformly continuous.