Analysis 2 – Week 1 Review

Abstract

These questions and remarks are intended to help you to review this week's contents; please take the time to do that, come up with examples and play around with the concepts. If you encounter problems, take actions soon. Work through your notes, ask me questions. Whenever you think about something that you don't understand make a note so that you do not forget. You can discuss the questions with me, your small group tutor, and class-mates. Such question lists will help you to stay on top of things by asking pointed questions.

1. Watch the youtube clip When Pi is Not $3.14\ldots$ on the channel Infinite Series. It is also linked next to Problem Sheet 1.

2. We started by reminding ourselves of the definition of convergence of a sequence $(a_n) \subseteq \mathbb{R}$: A sequence is convergent if and only if

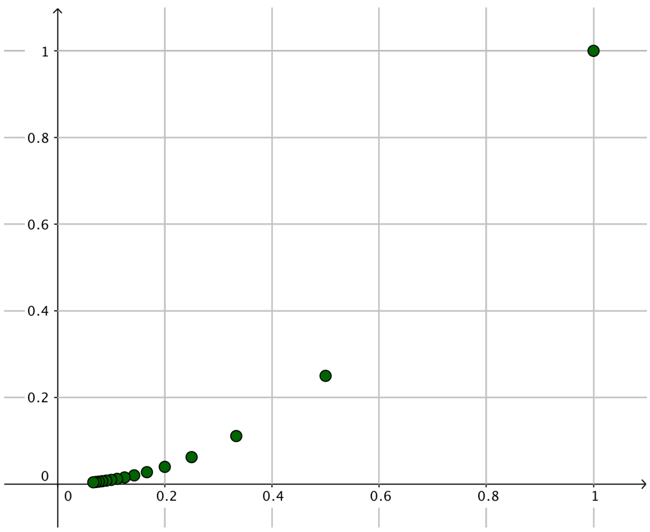
$$\exists a \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N}, n \geq n_0 \ \Rightarrow |a_n - a| < \varepsilon.$$

3. We then thought about sequences $(a_n) \subseteq \mathbb{R}^2$, i.e.

$$a_n = \begin{bmatrix} a_1^{(n)} \\ a_2^{(n)} \end{bmatrix}, \quad (a_n) = \left(\begin{bmatrix} a_1^{(1)} \\ a_2^{(1)} \end{bmatrix}, \begin{bmatrix} a_1^{(2)} \\ a_2^{(2)} \end{bmatrix}, \begin{bmatrix} a_1^{(3)} \\ a_2^{(3)} \end{bmatrix}, \begin{bmatrix} a_1^{(4)} \\ a_2^{(4)} \end{bmatrix}, \ldots \right).$$

An easy example could be $a_1^{(n)} := \frac{1}{n}$ and $a_2^{(2)} := \frac{1}{n}$, i.e.

$$a_n = \begin{bmatrix} a_1^{(n)} \\ a_1^{(n)} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n^2} \end{bmatrix}$$



4. Play a bit with GeoGebra and get a feeling for what is happening. The command

   Sequence((1/i, 1/i^2), i, 1, 15)

   produces the above sequence.

5. We introduced different definitions of convergence and claimed that they are all equivalent. The two concrete ones were

(a) A sequence $(a_n) \subseteq \mathbb{R}^2$ converges if and only if there exists an $a = [a_1\,,\, a_2]^T \in \mathbb{R}^2$ such that for all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$ the inequality

$$\sqrt{(a_1^{(n)} - a_1)^2 + (a_2^{(n)} - a_2)^2} < \varepsilon$$

holds.

(b) A sequence $(a_n) \subseteq \mathbb{R}^2$ converges if and only if there exists an $a = [a_1\,,\, a_2]^T \in \mathbb{R}^2$ such that for all $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_0$ the inequality

$$\left|a_1^{(n)} - a_1\right| + \left|a_2^{(n)} - a_2\right| < \varepsilon$$

holds.

We discussed that we can show that these two definitions are equivalent if we can prove that there exist constants $C_1, C_2 > 0$ such that

$$\left|a_1^{(n)} - a_1\right| + \left|a_2^{(n)} - a_2\right| \leq C_1\sqrt{(a_1^{(n)} - a_1)^2 + (a_2^{(n)} - a_2)^2}$$

and

$$\sqrt{(a_1^{(n)} - a_1)^2 + (a_2^{(n)} - a_2)^2} \leq C_2 \left|a_1^{(n)} - a_1\right| + \left|a_2^{(n)} - a_2\right|$$

hold true. Can you clearly explain why these inequalities would be helpful? Assuming that they hold, can you give a proof of the equivalence of the two notions of convergence above?

6. In the very first lecture, we added the claim that a sequence (a_n) is convergent if and only if, the component sequences are convergent.
This statement is also equivalent to the others we made in the lecture, in particular equivalent to Definition 1.6. and Definition 1.7. One direction is easy. Assume that the sequence $(a_n) \subseteq \mathbb{R}^2$ converges with respect to either of the definitions above. We clearly have

$$|a_1^{(n)} - a_1| = \sqrt{|a_1^{(n)} - a_1|^2} \leq \sqrt{|a_1^{(n)} - a_1|^2 + |a_2^{(n)} - a_2|^2}$$

since $|a_2^{(n)} - a_2|^2 \geq 0$ and the fact that the square root is a monotonically increasing function. The same way, we have

$$|a_2^{(n)} - a_2| \leq \sqrt{|a_1^{(n)} - a_1|^2 + |a_2^{(n)} - a_2|^2}.$$

Additionally, the inequalities

$$\begin{aligned} |a_1^{(n)} - a_1| &\leq |a_1^{(n)} - a_1| + |a_2^{(n)} - a_2|, \\ |a_2^{(n)} - a_2| &\leq |a_1^{(n)} - a_1| + |a_2^{(n)} - a_2| \end{aligned}$$

are immediate. Thus $(a_1^{(n)}), (a_2^{(n)}) \subseteq \mathbb{R}$ are convergent if (not yet and only if) the sequence $(a_n) \subseteq \mathbb{R}^2$ converges according to either of the two above definitions. The converse will be done next week. You can try to think about what we would have to do. In principal, I explained it when I was talking about the pictures I brought to class on Tuesday.

7. Use again GeoGebra to find examples for the following:

- How many ways can you make a sequence diverge? For example, does the sequence

$$a_n = \begin{bmatrix} n \\ \frac{1}{n} \end{bmatrix}$$

converge? (Plot it.)

- What does a not convergent but bounded sequence look like?
- Can you find sequences having $[1\,,\, 1]^T$ as a limit?