Analysis 2 – Week 4 Review

Abstract

These questions and remarks are intended to help you to review this week's contents; please take the time to do that, come up with examples and play around with the concepts. If you encounter problems, take actions soon. Work through your notes, ask me questions. Whenever you think about something that you do not understand make a note so that you do not forget. You can discuss the questions with me, your small group tutor, and class-mates. Such question lists will help you to stay on top of things by asking pointed questions.

1. Write down the definitions of *open cover*, *finite sub-cover*, and *compact set*.

2. Consider the following sets and the given covers. Which ones admit finite sub-covers? If they do, write one down.

   - Consider $(0, 1)$ and the open cover $\{\mathcal{O}_\varepsilon \,:\, \varepsilon \in (0, 1)\}$ with $\mathcal{O}_\varepsilon := (\varepsilon, 1)$. What about $\{\mathcal{O}_\varepsilon : \varepsilon \in (0, 1]\}$ with $\mathcal{O}_\varepsilon = (-1 + \varepsilon, 1 - \varepsilon)$.
   - Consider $K = [0, 1] \times [0, 1]$ and the open cover $\{\mathcal{O}_x \,:\, x \in K\}$[1] with

   $$\mathcal{O}_x = B_{\frac{2}{3}}(x) = \left\{ y \in \mathbb{R}^2 \,:\, \rho_2(x, y) < \frac{2}{3} \right\}.$$

   - Consider $\mathbb{R}$ and the open cover $\{\mathcal{O}_R \,:\, R \in (0, +\infty)\}$, where $\mathcal{O}_R := (-R, R)$.

3. Go through the proof of the fact that $[a, b]$ is a compact set. Do you understand all arguments? Can you give a rough sketch of the proof yourself?
   Read the second proof provided on LEARN. Which one do you like better?

4. Show in full detail that $\emptyset$ is compact.

5. Show in full detail that a finite sub-set of $\mathbb{R}^d$ is compact.

6. What are important properties of compact sets?

7. What does the Theorem of Heine–Borel state?

8. Decide which of the following sets are compact and explain your reasoning.

   (a) $\overline{B}_1(0) = \{y \in \mathbb{R}^d \,:\, \rho_2(0, y) \leq 1\}$,
   (b) $[0, 1]^3 = [0, 1] \times [0, 1] \times [0, 1] = \{(x, y, z) \in \mathbb{R}^3 \,:\, x, y, z \in [0, 1]\}$,
   (c) $S_1(0) = \{y \in \mathbb{R}^d \,:\, \rho_2(0, y) = 1\}$.

9. To conclude your review of Chapter 2, make a chart of all the notions introduced and their relations.

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[1]The index set of the open cover is the set of points itself.