Analysis 2 – Week 5 Review

Abstract

These questions and remarks are intended to help you to review this week's contents; please take the time to do that, come up with examples and play around with the concepts. If you encounter problems, take actions soon. Work through your notes, ask me questions. Whenever you think about something that you do not understand make a note so that you do not forget. You can discuss the questions with me, your small group tutor, and class-mates. Such question lists will help you to stay on top of things by asking pointed questions.

1. This week, we covered the (ε, δ)-definition of limits of functions and some consequences. In total, we had one definition ant=d two theorems. The rest of the discussion in the notes was centred around motivating the definition and theorems an fleshing them out. Thus, reviewing you should start with the definition and

   (a) write it down in the particular case $m = d = 1$,

   (b) draw a picture of that situation,

   (c) give examples of the special and the general case,

   (d) write down the negation of the definition.

   Then go to the two theorems and start working on Problem Sheet 3.

2. If you have difficulties to digest the definition, I suggest to have a look at the following two videos which are good treatments and are of higher quality than the random youtube maths video:

   - Limits of Functions (The Epsilon/Delta Definition) - Part 1 of 2
   - Limits of Functions (The Epsilon/Delta Definition) - Part 2 of 2

3. You can try to transport some proofs from sequences to the new scenario using the sequences characterisation of limits of functions.[1]

   (a) Assume that $(a_n) \subseteq \mathbb{R}$ and that $(a_n) \to a$ for some $a \in \mathbb{R}$ with $a > 0$. Prove that there exists $n_0 \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq n_0$.

   (b) From that we get immediately the following result: Let $f : (a, b) \to \mathbb{R}$ and let $y_0 \in \mathbb{R}$, $y_0 > 0$ with
   $$\lim_{x \to x_0} f(x) = y_0.$$
   Then, For all sequences $(x_n) \subseteq (a, b) \setminus \{x_0\}$, there exists $n_0 \in \mathbb{N}$ such that $f(x_n) > 0$ for all $n \geq n_0$.

   (c) Can we conclude the following?

   Let $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. Suppose that
   $$\lim_{x \to x_0} f(x) = y_0$$
   for some $y_0 \in \mathbb{R}$, $y_0 > 0$. Then, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in [x_0 - \delta, x_0 + \delta] \setminus \{x_0\}$.

[1] This is the general reason why we would try to connect new notions with old ones. First it may help us understand the situation but it will also allow us to transport proof strategies from one place to another.

Can you use a strategy employed in the proof of Theorem 3.1 in Chapter 3? If you did Problem 3 on Problem Sheet 3 with an (ε, δ)-argument, can you redo it in the spirit of what I described here? What result of Analysis 1 could be helpful? Sequences and boundedness ...?

4. The other way around, can you prove the result in 3(c) with an (ε, δ)-argument? (Hint: Draw a picture and think about what you can do with $\varepsilon = \frac{y_0}{2} > 0$.)