# Analysis 2 – Week 7 Review

**Abstract**

These questions and remarks are intended to help you to review this week's contents; please take the time to do that, come up with examples and play around with the concepts. If you encounter problems, take actions soon. Work through your notes, ask me questions. Whenever you think about something that you do not understand make a note so that you do not forget. You can discuss the questions with me, your small group tutor, and class-mates. Such question lists will help you to stay on top of things by asking pointed questions.

This weeks review remarks:

1. This week, we learned that continuous functions as locally sign preserving, locally bounded, and have the amazing intermediate value property which we already used successfully to proof
   - that odd degree polynomials have at least one real root,
   - that there exists the $n$th root of a positive real number,
   - that certain equations have solutions, and
   - that certain continuous functions are surjective.

2. Can you reproduce the proofs of the statements that continuous functions are locally bounded and locally sign preserving? Can you prove the result using a corresponding sequence result via contraposition as I discussed in class? See also the exam information file on LEARN.

3. The proof of the intermediate value theorem proceeded by partitioning the interval $[a, b] = X_+ \cup X_0 \cup X_-$, where

$$\begin{aligned} X_+ &:= \{x \in [a, b] : \ f(x) > 0\}, \\ X_0 &:= \{x \in [a, b] : \ f(x) = 0\}, \\ X_- &:= \{x \in [a, b] : \ f(x) < 0\}, \text{ and} \end{aligned}$$

   then showing that $X_0 \neq \emptyset$. Can you supply the details?

   Note that the proof depend on the completeness axiom and try to identify where the proof fails if the function is not defined on an interval $[a, b]$ but on $[a, b] \cap \mathbb{Q}$.

4. The intermediate value theorem allows to prove the following: Given a plate with mashed potatoes, we can always find a way to separate it 50/50 by just one cut.

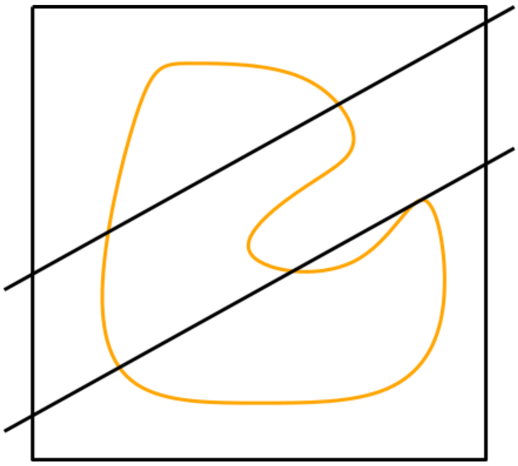

Figure 1: Half a plate of mashed potato.

Can you explain how it works? The first line has more than 50% of mashed potato below, the second one more than 50% above. Thus, there is a position on between where you have exactly 50/50. Where doe the continuity come in? How exactly does one apply the intermediate value theorem? (We assume that all terms like the *mass* of the mashed potatoes are well defined and have, from the heavens, some function that gives us what we need.)

5. As said, we have a plate with mashed potatoes and baked beans. At any particular angle $\alpha$, we can half the mashed potatoes. Then, there is either more or less of baked beans to the left of the cut, say less. Now, we vary $\alpha$ in a continuous fashion by $\pi$ until the knife has turned. At each $\alpha$, we make sure to half the potatoes. Now more than half the baked beans are on the left. Thus, we passed through an intermediate position where both, baked beans and mashed potatoes, were divided fairly 50/50.

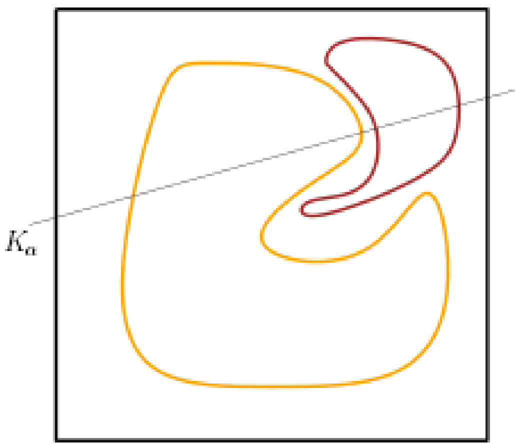


Figure 2: Half a plate of mashed potato with baked beans.

6. If you are interested read the Wikipedia article on the Ham Sandwich Theorem and follow some references.

7. Another interesting theorem of this week was Brouwer's Fixed Point Theorem which we proved with the Intermediate Value Theorem. The video The Brouwer Fixed Point Theorem a nice explanation of Brouwer's Fixed Point Theorem.[1]

8. Here are some further good resources concerning fixed points after you understood the 1D case. They should be seen as further education.

   (a) The Vsauce video on Fixed Points is of philosophical nature and discusses fixed points in general as well as Brouwer's Fixed point theorem.

   (b) The Infinite Series video on Brouwer's Fixed Point Theorem is more advanced but gives you a gimps in further possible developments and the amazing interconnectedness of the fields of mathematics.

   (c) The 3Blue1Brown video Who (else) cares about topology? Stolen necklaces and Borsuk-Ulam contains a discussion of the Borsuk-Ulam Theorem which is related to Fixed Points. See also here.

[1] The audio is not very good.