Analysis 2 – Week 8 Review

Abstract

These questions and remarks are intended to help you to review this week's contents; please take the time to do that, come up with examples and play around with the concepts. If you encounter problems, take actions soon. Work through your notes, ask me questions. Whenever you think about something that you do not understand make a note so that you do not forget. You can discuss the questions with me, your small group tutor, and class-mates. Such question lists will help you to stay on top of things by asking pointed questions.

This weeks review remarks:

1. We defined a function $f : (a, b) \to \mathbb{R}$ as being differentiable at $x_0 \in (a, b)$ if and only if the limit
$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$
exists. We denoted the limit by $f'(x_0)$ in that case. Use the (ε, δ)-definition of limits to formulate when a function $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$. An illustration of the (ε, δ)-definition can be found here. Here is a clip from a BBC documentary illustrating the definition of derivative. (You might need your Athens password.)

2. Find examples (you may choose (a, b) as needed) of functions $f : (a, b) \to \mathbb{R}$ such that

   (a) The limit (1) exists for all $x_0 \in (a, b)$.

   (b) There is an $x_0 \in (a, b)$ such that

      (i) $\lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h}$ exists and

      (ii) $\lim_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h}$ does not exist

   (One-sided limits were discussed in Section 3.5 in the lecture notes.)

   (c) There is an $x_0 \in (a, b)$ such that

      (i) $\lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h}$ exists and

      (ii) $\lim_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h}$ exits

   but they are not equal.

   The examples should be easier to find if you think by drawing pictures instead of formulas. Come up with them once you know how the graph looks like. Good functions to start with are $f(x) = |x|$ and $f(x) = \sqrt{x}$ and then graphs that somehow combine those with some new elements as needed.

3. Can you explain (1) to a fellow student with a picture? You can use GeoGebra to make pictures. Here is a prepared worksheet.

4. To get fluent in the application of the definitions, work through the problems concerning differentiability on Problem Sheet 6, especially Section 2.

5. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and compute $\phi(h)$ according to Problem 6 on Problem Sheet 6. Do the same for $f : (0, +\infty) \to (0, +\infty)$, $f(x) = \frac{1}{x}$. Write formulae for $f(x_0 + h)$ for x_0 in the appropriate domain. Can you interpret $\phi(h)$?