Contents

1 Organisational information, general outline

The final exam is at **8 June** in the afternoon. Make sure that you are well rested and have eaten properly, i.e. not nothing and not too much. There is no use in trying to cramp something in the night before. Get sleep. What you have not learned till then you will not learn that day but will in fact inhibit your retrieval ability through sleep deprivation!

The exam is divided in two parts. The first part is worth 40 points and will cosiest of stating definitions and theorems and giving proofs. The second part, also worth 40 points, is a set of 16 multiple choice questions (2.5 points each) where you have to choose 1 out of 4.

The structure of the exam is only relevant for you to plan your time efficiently. The MC questions are not intended to reduce the amount of learning but reduce marking time. There can and will be questions asking you about theorems and their consequences and if you do not know the statements, you can not answer them apart from guessing; e.g.

> Pick the right one:[1]
> (a) A continuous function $f : \mathbb{R} \to \mathbb{R}$ has always a fixed point.
> (b) A linear map $f : \mathbb{R}^2 \to \mathbb{R}$ has always a fixed point.
> (c) A continuous function $f : (-3, 1) \to [-3, 1]$ has always a fixed point.
> (d) A continuous function $f : [0, 1] \to [0, 2]$ has always a fixed point.

Thus, I urge you to not concentrate on the concrete test but on the understanding of the introduced concepts and their interrelations. Keep also in mind, that the contents of this lecture will be extended in Analysis 3 and other modules and if you do not learn them properly, you will have a hard time there as well. If you properly learn, they will be considerably easier. Consider the long run!

2 Tested knowledge (Module description)

Knowledge that will be tested: As the module description states, you should

[1] It is (B) should you be interested.

- recall the definition and state basic properties of continuous functions and of differentiable functions of one and several variables (Contents of Chapters 3, 4, 5, and 6.);
- recall the definition of the Riemann integral as a limit, and its basic properties (its existence, the Fundamental Theorem of Calculus, changes of variables) and explain its relationship to the area under a curve (Contents of Chapters 7, 8, and 9.).

Skills that will be tested: As the module description states, you should

- be able to reproduce proofs of the basic properties of functions of one variable, and to be able to construct simple proofs in similar style;
- be able to work with the Riemann integral as a limit, and appreciate that it exists for large classes of functions but not for every function.

3 Tested knowledge (A bit more specific)

In principal you must be able to state any definition given in the lecture. The simple reason is that we would like to ensure that you know what you are talking about.
You must state the definition with explanations where necessary. For example, if you are asked to define what it means that a function f is differentiable and what the derivative is the an appropriate answer would be: *A function $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) iff for all $x \in (a, b)$ the limit*

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists. The limit is then denoted by $f'(x)$ and the derivative of f is the map $x \mapsto f'(x)$.
Obviously one could ask for differentiability only at a point and then the answer would be slightly different. **Please read the question carefully.** You will not get points for correct answers to questions I did not ask. Another example is the following. If you are asked to define step functions, it is not sufficient to write

$$g(x) = \sum_{i=1}^{n} c_i \chi_{I_i}(x).$$

Instead you should write: *A function $g : I \to \mathbb{R}$ is a step-function if it can be written as*

$$g(x) = \sum_{i=1}^{n} c_i \chi_{I_i}(x),$$

where $c_i \in \mathbb{R}$ and $I_i \subseteq I$ are intervals. The functions χ_{I_i} are the indicator functions of the I_i.
The exam will ask for only a few definitions for you to write down explicitly but you will have to know them to answer the rest of the questions. For example if one asks to rewrite the condition for differentiability as an (ε, δ)-criterion, you will need to know the (ε, δ)-definition of a limit even though I did not ask for it specifically. Also the multiple

choice questions may contain definition questions where you have to choose the correct one among different versions. The multiple choice questions are also such that you need to know the theorems that relate the definitions even if you do not have to state them all. Please refer to the Yes/No questions as preparation for the multiple choice questions.

There will be no harder proofs than on the practice exam for you to perform. Thus, Fermat's theorem, Rolle's theorem, Brouwer's Fixed Point Theorem, the Mean Value Theorem, and the two auxiliary lemmas (Lemmas 4.1 and 4.2) for the EVT and IVT should be no problem to prove. The following question should not cause goose bumps

Question Let $f \in C[a,b]$, $f(x) \geq 0$ for $x \in [a,b]$, $x_0 \in (a,b)$, and $f(x_0) > 0$. Prove that

$$\int_a^b f(x)dx > 0.$$

As the word *prove* indicates, you should write details. Also, proofs have words, they are not just decoration in the Lecture Notes.

Concerning integration theory, you should know whether integrals like

$$\int_1^{+\infty} \frac{1}{\sqrt{x}}dx, \quad \int_0^1 \frac{1}{x}dx, \quad \int_{-\infty}^{+\infty} \frac{1}{1+x^2}dx$$

exist or not. (See also Review Sheet 11 and Problem Sheet 10.) You should be able to rigorously (see above) state the definitions of the three integrals we defined:

1. Integral of a step function,
2. Integral for regulated functions, and
3. Riemann Integrability.

Finally, you should be able to state the Fundamental Theorem of Calculus and prove its statement or some weaker form as that $F(x) = \int_a^x f(y)dy$ is continuous if $f \in C[a,b]$. You should also be able to construct examples of the following kind: Find a sequence of functions $(f_n)_{n\in\mathbb{N}} \subseteq C[a,b]$ with $f_n \to 0$ pointwise but

$$\lim_{n\to\infty} \int_a^b f_n(x)dx \neq 0.$$

I would accept pictures as long as they are done with some care and all necessary information is present., i.e. explicit sequences etc pp. It must be easy to see that the convergence is correct and the integral condition holds.

Compactness is relevant insofar as you should know the definition and a characterisation (Heine-Borel). Its most important application for us is that on compact sets continuity and uniform continuity are equivalent.

4 Things I will not ask

1. I will not ask you specifically for the definition of a norm or a metric. However, as you have seen during the semester, you must be able to operate with them as they are the basis of limits, continuity and differentiability.
2. I will not ask you the definitions of vector spaces, algebras or linear maps. However, you need to be able to work with them, i.e. you should know that linear combinations of step-functions are step-functions etc pp.
3. The parts that are indicated in the lecture notes as not examinable.
4. Multidimensional integrals. As a mathematician, you should understand how these things work but if you are behind anyway, they are no priority.
5. The proofs of the IVT and EVT. Though you should understand the mechanisms as the techniques can be used in other contexts.
6. Proofs of the Theorems of Heine-Borel and Heine.
7. Proof of the partial integration formula or change of variables formula in the chapter of Integration.
8. You should be able to tell whether the following integrals would converge (absolutely) but you wont have to calculate them:

$$\int_0^1 \frac{1}{\sqrt[5]{x}}dx, \quad \int_0^{+\infty} \frac{1}{1+x}dx, \quad \int_0^{+\infty} \frac{\sin(x)}{1+x^4}dx, \quad \int_1^{+\infty} \frac{1}{\sqrt{1+x^2}}dx.$$

5 A couple of tips

1. Start immediately.
2. Go through the Check List online to see where you stand and what you should do. Start with the missing basics.
3. Use pen and paper. Writing improves retention!
4. *Improve you retrieval! Take a blank sheet and write down everything you remember about a certain topic as continuity or differentiability. That help to strengthen your memory and trains retrieval.*
5. When you write down a definition, you should write down some examples and non-examples. Make sure you understand why they are examples and non examples. Draw pictures if appropriate. For example, $\mathbb{R}^n$ is an open set, do I understand why? Do I understand why a set $\{x\}$ for $x \in \mathbb{R}^n$ is not open?
6. Try to find a colloquial definition where possible. In an open set, for example, no point is ever alone, all are surrounded by friends. In a closed set, there are possibly points which are not surrounded by friends. In an open set, there are no loners at all. Work from there to make the definition more precise.

7. Try to compare definitions that talk about similar things. Like convergence of sequences of real numbers and convergence of sequences of functions, pointwise and uniform.

8. Speak definitions out loud. Speak out loud expressions you write with quantifiers.

9. Make breaks! After a couple of hours study, do something that makes you happy. Have a walk with your sweetheart, watch a film, have some cocoa, whatever.

10. Do not simply despair. You are not alone. Meet with friends twice a week and discuss definitions/theorems you don't understand. Explain, think, rethink, explain again.

    > Ever tried. Ever failed. No matter. Try again. Fail again. Fail better.
    > Samuel Beckett

11. Accept some advise. Some methods that have worked in school are really not gonna work this time.

    (a) Prepping for Finals - How to Make a Study/Revision Plan

    (b) Improve your Study Skills with these Tips!

    You could also ask people in the MEC as many of them are Psychologists and can give you some tips.

12. Ask when you have a question! Of course I expect you to ponder a question hard to try to get ans answer by yourself and, obviously, that you browse the Lecture Notes for a solution. If that does not help jump over your shadow and ask me, people in the MLSC, friends.

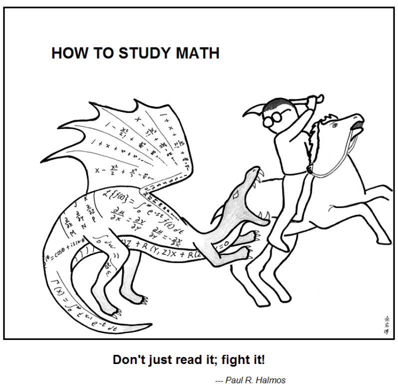

Figure 1: The picture is from Abstruse Goose.