Well–posedness of hyperbolic systems with multiplicities and smooth coefficients

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Abstract

We study hyperbolic systems with multiplicities and smooth coefficients. In the case of non-analytic, smooth coefficients, we prove well-posedness in any Gevrey class and when the coefficients are analytic, we prove C^∞ well-posedness. The proof is based on a transformation to block Sylvester form introduced by D'Ancona and Spagnolo in [2] which increases the system size but does not change the eigenvalues. This reduction introduces lower order terms for which appropriate Levi-type conditions are found. These translate then into conditions on the original coefficient matrix. This paper can be considered as a generalisation of [1], where weakly hyperbolic higher order equations with lower order terms were considered.

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Introduction

We consider the Cauchy problem

$$\begin{cases} D_t u = A(t, D_x)u, & (t, x) \in [0, T] \times \mathbb{R}^n \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n \end{cases}, \tag{CP}$$

where $u = (u_1, \dots, u_m)^T$, and we assume that the (by $\langle\xi\rangle^{-1}$ rescaled) characteristic roots $\lambda_1(t, \xi), \dots, \lambda_m(t, \xi)$ of $A(t, \xi)$ are real (hyperbolicity) and that there exists a constant such that

$$\lambda_i(t, \xi)^2 + \lambda_j(t, \xi)^2 \le C(\lambda_i(t, \xi) - \lambda_j(t, \xi))^2, \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}^n \tag{Roots}$$

for all $1 \le i < j \le m$.

We prove two well-posedness results for problem (CP). The first one is a well posedness result in every Gevrey spaces under the assumption that the coefficients of $A(t, \xi)$ are $C^\infty[0, T]$ with respect to t and the second one is a C^∞-well-posedness result under the assumption that the coefficients of $A(t, \xi)$ are $C^\omega[0, T]$ with respect to t. To make the statements of the results more accessible, we first state the scheme of the proof in the next section. The main point will be to find appropriate Levi-type conditions for a new system derived from (CP).

Scheme of the proof

Step 1 Compute the adjunct matrix $\mathbf{adj}(I_m\tau - A(t, \xi)) = \mathbf{cof}(I_m\tau - A^T(t, \xi))$, where I_m is the identity matrix of size $m \times m$:

$$\mathbf{adj}(I_m D_t - A(t, D_x)) = \sum_{h=0}^{m-1} \mathbf{A}_h(t, D_x) D_t^{m-1-h},$$

where

$$\mathbf{A}_h(t, D_x) = \sum_{h'=0}^{h} \sigma_{h'}^{(m)}(\lambda) A^{h-h'}(t, D_x).$$

The $\sigma_{h'}^{(m)}(\lambda)$ are given by

$$\sigma_h^{(m)}(\lambda) = (-1)^h \sum_{1 \le i_1 < \dots < i_h \le m} \lambda_{i_1} \dots \lambda_{i_h}$$

for all $1 \le h \le m$ and $\sigma_0^{(m)}(\lambda) = 1$. Then, we have the relation

$$\mathbf{adj}(I_m\tau - A(t, \xi))(I_m\tau - A(t, \xi)) = \det(I_m\tau - A(t, \xi)) = \sum_{h=0}^{m} c_h(t, \xi) I_m \tau^{m-h},$$

where the $c_h(t, \xi)$ are homogeneous polynomials of order h in ξ and are given by the coefficients of the characteristic polynomial of $A(t, \xi)$.

Step 2 Apply the operator $\mathbf{adj}(I_m D_t - A(t, D_x))$, associated to the symbol $\mathbf{adj}(I_m\tau - A(t, \xi))$, to the system (CP) and obtain a set of scalar equations for u_1 to u_m, where the operator acting on these is associated to $\det(I_m\tau - A(t, \xi))$. Additionally, we get the lower order terms

$$B(t, D_t, D_x)u = \left(- \sum_{h=0}^{m-2} \sum_{j=1}^{m} b_{ij}^{(h+1)}(t, D_x) D_t^h u_j \right)_{i=1,\dots,m},$$

where the $b_{ij}^{(h+1)}(t, \xi)$ are the ij-elements of the matrix $\mathbf{B}_{h+1}(t, \xi)$:

$$\mathbf{B}_{h+1}(t, D_x) = \sum_{h'=0}^{m-2-h} \binom{m-1-h'}{h} \mathbf{A}_{h'}(t, D_x)(D_t^{m-1-h-h'} A)(t, D_x),$$

$$\mathbf{A}_h(t, D_x) = \sum_{h'=0}^{h} \sigma_{h'}^{(m)}(\lambda) A^{h-h'}(t, D_x).$$

Step 3 Convert the resulting set of equations

$$\det(I_m D_t - A(t, D_x))u + l.o.t. = 0$$

to Sylvester block diagonal form following the method of Taylor, i.e by setting

$$U = (U_1, U_2, \dots, U_m)^T, \text{ where}$$
$$U_k = \left(\langle D_x\rangle^{m-1} u_k, D_t \langle D_x\rangle^{m-2} u_k, \dots, D_t^{m-1} u_k\right)$$

for $k = 1, \dots, m$. This transformation maps each equation to a system in Sylvester form and glues those systems in block diagonal form together. Hence, we achieve a block diagonal form with Sylvester blocks associated to the characteristic polynomial of (CP), i.e. $\mathcal{A}(t, D_x)$ is composed of m copies of

$$\langle D_x\rangle \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 \\ -c_m(t, D_x)\langle D_x\rangle^{-m} & -c_{m-1}(t, D_x)\langle D_x\rangle^{-m+1} & \dots & \dots & -c_1(t, D_x)\langle D_x\rangle^{-1} \end{pmatrix}.$$

The lower order terms, denoted by $\mathcal{B}(t, \xi)$ will be composed by m matrices of size $m \times m^2$ of the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{i,1}(t, D_x) & l_{i,2}(t, D_x) & \dots & \dots & l_{i,m^2-1}(t, D_x) & l_{i,m^2}(t, D_x), \end{pmatrix}$$

$i = 1, \dots, m$. As a general formula for the non-zero elements of $\mathcal{B}(t, D_x)$, we can write

$$l_{i,h+1+(j-1)m}(t, D_x) = b_{ij}^{(h+1)}(t, D_x)\langle D_x\rangle^{1-m+h}$$

for $j = 1, \dots, m$ and $h = 0, \dots, m-2$.

Step 4 Consider the resulting system

$$\begin{cases} D_t U = \mathcal{A}(t, D_x)U + \mathcal{B}(t, D_x)U \\ U|_{t=0} = U_0 \end{cases}, \tag{SylvesterCP}$$

where $\mathcal{A}(t, D_x)$ and $\mathcal{B}(t, D_x)$ are matrices of size $m^2 \times m^2$ with a special structure. As explained above, $\mathcal{A}(t, D_x)$ is a block diagonal matrix with m identical blocks of Silvester matrices having the same eigenvalues as $A(t, \xi)$ and $\mathcal{B}(t, D_x)$ is composed of $m \times m^2$ blocks with only the last row not identically zero. Since the original homogeneous system has been transformed into a system with lower order terms, to get well-posedness of the corresponding Cauchy problem (SylvesterCP), we need to find some Levi-type conditions. These are obtained by following the ideas for scalar equations in [1].

Step 5 We apply the partial Fourier transform with respect to x to (SylvesterCP) and we prove an energy estimate from which the assertions of the well-posedness theorems follow in a standard way. A key point is the construction of the quasi-symmetriser of the matrix $\mathcal{A}(t, \xi)$; see [2].

Statements of the well-posedness results

We assume that the matrix $\mathcal{B}(t, \xi)$ in (SylvesterCP) satisfies

$$\sum_{k=1}^{m} |b_{kj}^{(l)}(t, \xi)|^2 \prec \sum_{i=1}^{m} |\sigma_{m-l}^{(m-1)}(\pi_i\lambda)|^2 \tag{Levi}$$

for any $l = 1, \dots, m-1$ and $j = 1, \dots, m$.

Theorem 1. *Let $A(t, D_x)$, $t \in [0, T]$, $x \in \mathbb{R}^n$, be an $m \times m$ matrix of first order differential operators with C^∞-coefficients. Let $A(t, D_x)$ have real eigenvalues satisfying condition* (Roots). *Let*

$$\begin{cases} D_t u - A(t, D_x)u = 0, & (t, x) \in [0, T] \times \mathbb{R}^n \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n \end{cases}$$

be the Cauchy problem (CP). *Assume that the Cauchy problem* (SylvesterCP), *obtained from* (CP) *by block Sylvester reduction and the lower order terms matrix $\mathcal{B}(t, D_x)$ fulfills the Levi-type conditions* (Levi). *Hence, for all $s \ge 1$ and for all $u_0 \in \gamma^s(\mathbb{R}^n)^m$ there exists a unique solution $u \in C^1([0, T], \gamma^s(\mathbb{R}^n))^m$ of the Cauchy problem* (CP).

The next theorem gives a condition that is easier to check and yields the assertion of the last theorem. However, in general, the condition is more restrictive.

Theorem 2. *Let $A(t, D_x)$, $t \in [0, T]$, $x \in \mathbb{R}^n$, be an $m \times m$ matrix of first order differential operators with C^∞-coefficients. Let A have real eigenvalues satisfying condition* (Roots) *and let $Q = (q_{ij})$ be the symmetriser of $A_0 = \langle\xi\rangle^{-1}A$. Assume that*

$$\max_{k=1,\dots,m-1} \|D_t^k A_0(t, \xi)\|^2 \prec q_{j,j}(t, \xi) \tag{Levi II}$$

for all $(t, \xi) \in [0, T] \times \mathbb{R}^n$ and $j = 1, \dots, m-1$. Hence, for all $s \ge 1$ and for all $u_0 \in \gamma^s(\mathbb{R}^n)^m$ there exists a unique solution $u \in C^1([0, T], \gamma^s(\mathbb{R}^n))^m$ of the Cauchy problem (CP).

The last theorem states C^∞–well posedness under analyticity assumptions.

Theorem 3. *If all entries of $A(t, D_x)$ in* (CP) *are analytic on $[0, T]$, the eigenvalues satisfy* (Roots) *and the entries of the matrix $\mathcal{B}(t, \xi)$ in* (SylvesterCP) *satisfy the Levi conditions* (Levi) *for ξ away from 0, then the Cauchy problem* (CP) *is C^∞ well-posed, i.e., for all $u_0 \in C^\infty(\mathbb{R}^n)^m$ there exists a unique solution $u \in C^1([0, T], C^\infty(\mathbb{R}^n))^m$ of the Cauchy problem* (CP).

Examples

Example 1. *When $m = 2$, the Levi-type conditions* (Levi) *imply* (Levi II). *Indeed, the Levi-type conditions in that case are formulated as*

$$(|D_t a_{11}(t)|^2 + |D_t a_{21}(t)|^2)\langle\xi\rangle^{-2} \prec \lambda_1^2(t, \xi) + \lambda_2^2(t, \xi),$$
$$(|D_t a_{12}(t)|^2 + |D_t a_{22}(t)|^2)\langle\xi\rangle^{-2} \prec \lambda_1^2(t, \xi) + \lambda_2^2(t, \xi).$$

Example 2. *In the special case $D_t^2 u - a(t)D_x^2 u = 0$ with $a(t) \ge 0$ and appropriate Cauchy data, the Levy-type condition is automatically satisfied for $a \in C^2[0, T]$. Indeed, with $a_{11} = 0$, $a_{12} = 1$, $a_{21} = a(t)$, and $a_{22} = 0$, condition* (Levi II) *becomes $|D_t a(t)|^2 \le Ca(t)$ which is satisfied by Glaeser's inequality.*

Example 3. *Let us now take a 3×3 matrix A with trace zero. For simplicity let us assume that $n = 1$ and that the eigenvalues of the corresponding A_0 are $\lambda_1(t, \xi) = -\sqrt{a(t)}\xi\langle\xi\rangle^{-1}$, $\lambda_2(t, \xi) = 0$ and $\lambda_3(t, \xi) = \sqrt{a(t)}\xi\langle\xi\rangle^{-1}$ with $a(t) \ge 0$ for $t \in [0, T]$. It follows that the hypothesis* (Roots) *on the eigenvalues is satisfied. By direct computation we get*

$$q_{1,1} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 = a(t)\xi^2\langle\xi\rangle^{-2},$$
$$q_{2,2} = (\lambda_1 + \lambda_2)^2 + (\lambda_1 + \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 = 2a(t)\xi^2\langle\xi\rangle^{-2}.$$

It follows that both $q_{1,1}$ and $q_{2,2}$ are comparable to a and therefore, we conclude that

$$|b_{kj}^{(1)}|^2 = |D_t^2 a_{kj} + 2D_t a_{kj}|^2 \prec a(t),$$
$$|b_{kj}^{(2)}|^2 = |a_{k1}D_t a_{1j} + a_{k2}D_t a_{2j} + a_{k3}D_t a_{3j}|^2 \prec a(t),$$

for $k = 1, 2, 3$ and $j = 1, 2$. We can easily see on the matrix

$$A(t, \xi) = \begin{pmatrix} 0 & a(t) & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xi$$

that the conditions above on the entries of A entail

$$|D_t^k a(t)|^2 \prec a(t)$$

for all $t \in [0, T]$ and $k = 1, 2$, i.e. condition (Levi II).

References

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