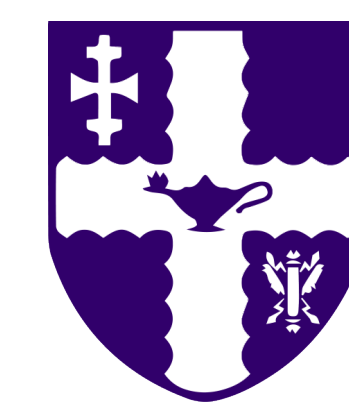


Well-posedness of hyperbolic systems with multiplicities and smooth coefficients

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Abstract

We study hyperbolic systems with multiplicities and smooth coefficients. In the case of non-analytic, smooth coefficients, we prove well-posedness in any Gevrey class and when the coefficients are analytic, we prove C^∞ well-posedness. The proof is based on a transformation to block Sylvester form introduced by D'Ancona and Spagnolo in [2] which increases the system size but does not change the eigenvalues. This reduction introduces lower order terms for which appropriate Levi-type conditions are found. These translate then into conditions on the original coefficient matrix. This paper can be considered as a generalisation of [1], where weakly hyperbolic higher order equations with lower order terms were considered.

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Introduction

We consider the Cauchy problem

$$\begin{cases} D_t u = A(t, D_x)u, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n \end{cases}, \quad (\text{CP})$$

where $u = (u_1, \dots, u_m)^T$, and we assume that the (by $\langle \xi \rangle^{-1}$ rescaled) characteristic roots $\lambda_1(t, \xi), \dots, \lambda_m(t, \xi)$ of $A(t, \xi)$ are real (hyperbolicity) and that there exists a constant such that

$$\lambda_i(t, \xi)^2 + \lambda_j(t, \xi)^2 \leq C(\lambda_i(t, \xi) - \lambda_j(t, \xi))^2, \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}^n \quad (\text{Roots})$$

for all $1 \leq i < j \leq m$.

We prove two well-posedness results for problem (CP). The first one is a well posedness result in every Gevrey spaces under the assumption that the coefficients of $A(t, \xi)$ are $C^\infty[0, T]$ with respect to t and the second one is a C^∞ -well-posedness result under the assumption that the coefficients of $A(t, \xi)$ are $C^\omega[0, T]$ with respect to t . To make the statements of the results more accessible, we first state the scheme of the proof in the next section. The main point will be to find appropriate Levi-type conditions for a new system derived from (CP).

Scheme of the proof

Step 1 Compute the adjunct matrix $\text{adj}(I_m D_t - A(t, \xi)) = \text{cof}(I_m D_t - A(t, \xi))$, where I_m is the identity matrix of size $m \times m$:

$$\text{adj}(I_m D_t - A(t, \xi)) = \sum_{h=0}^{m-1} \mathbf{A}_h(t, D_x) D_t^{m-1-h},$$

where

$$\mathbf{A}_h(t, D_x) = \sum_{h'=0}^h \sigma_{h'}^{(m)}(\lambda) A^{h-h'}(t, D_x).$$

The $\sigma_{h'}^{(m)}(\lambda)$ are given by

$$\sigma_h^{(m)}(\lambda) = (-1)^h \sum_{1 \leq i_1 < \dots < i_h \leq m} \lambda_{i_1} \dots \lambda_{i_h}$$

for all $1 \leq h \leq m$ and $\sigma_0^{(m)}(\lambda) = 1$. Then, we have the relation

$$\text{adj}(I_m D_t - A(t, \xi))(I_m D_t - A(t, \xi)) = \det(I_m D_t - A(t, \xi)) = \sum_{h=0}^m c_h(t, \xi) I_m \tau^{m-h},$$

where the $c_h(t, \xi)$ are homogeneous polynomials of order h in ξ and are given by the coefficients of the characteristic polynomial of $A(t, \xi)$.

Step 2 Apply the operator $\text{adj}(I_m D_t - A(t, D_x))$, associated to the symbol $\text{adj}(I_m \tau - A(t, \xi))$, to the system (CP) and obtain a set of scalar equations for u_1 to u_m , where the operator acting on these is associated to $\det(I_m \tau - A(t, \xi))$. Additionally, we get the lower order terms

$$B(t, D_t, D_x)u = \left(- \sum_{h=0}^{m-2} \sum_{j=1}^m b_{ij}^{(h+1)}(t, D_x) D_t^h u_j \right)_{i=1, \dots, m},$$

where the $b_{ij}^{(h+1)}(t, \xi)$ are the ij -elements of the matrix $\mathbf{B}_{h+1}(t, \xi)$:

$$\mathbf{B}_{h+1}(t, D_x) = \sum_{h'=0}^{m-2-h} \binom{m-1-h'}{h} \mathbf{A}_{h'}(t, D_x) (D_t^{m-1-h-h'} A)(t, D_x),$$

$$\mathbf{A}_h(t, D_x) = \sum_{h'=0}^h \sigma_{h'}^{(m)}(\lambda) A^{h-h'}(t, D_x).$$

Step 3 Convert the resulting set of equations

$$\det(I_m D_t - A(t, D_x))u + l.o.t. = 0$$

to Sylvester block diagonal form following the method of Taylor, i.e by setting

$$U = (U_1, U_2, \dots, U_m)^T, \quad \text{where} \\ U_k = (D_x)^{m-1} u_k, D_t(D_x)^{m-2} u_k, \dots, D_t^{m-1} u_k$$

for $k = 1, \dots, m$. This transformation maps each equation to a system in Sylvester form and glues those systems in block diagonal form together. Hence, we achieve a block diagonal form with Sylvester blocks associated to the characteristic polynomial of (CP), i.e. $A(t, D_x)$ is composed of m copies of

$$\langle D_x \rangle \begin{pmatrix} 0 & & 1 & & 0 & \dots & 0 \\ 0 & & 0 & & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & & 0 & & 0 & \dots & 1 \\ -c_m(t, D_x) \langle D_x \rangle^{-m} & -c_{m-1}(t, D_x) \langle D_x \rangle^{-m+1} & \dots & \dots & -c_1(t, D_x) \langle D_x \rangle^{-1} & \dots & 0 \end{pmatrix}.$$

The lower order terms, denoted by $B(t, \xi)$ will be composed by m matrices of size $m \times m^2$ of the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{i,1}(t, D_x) & l_{i,2}(t, D_x) & \dots & \dots & l_{i,m^2-1}(t, D_x) & l_{i,m^2}(t, D_x) \end{pmatrix}$$

$i = 1, \dots, m$. As a general formula for the non-zero elements of $B(t, D_x)$, we can write

$$l_{i,h+1+(j-1)m}(t, D_x) = b_{ij}^{(h+1)}(t, D_x) \langle D_x \rangle^{1-m+h}$$

for $j = 1, \dots, m$ and $h = 0, \dots, m-2$.

Step 4 Consider the resulting system

$$\begin{cases} D_t U = \mathcal{A}(t, D_x)U + \mathcal{B}(t, D_x)U, \\ U|_{t=0} = U_0 \end{cases}, \quad (\text{SylvesterCP})$$

where $\mathcal{A}(t, D_x)$ and $\mathcal{B}(t, D_x)$ are matrices of size $m^2 \times m^2$ with a special structure. As explained above, $\mathcal{A}(t, D_x)$ is a block diagonal matrix with m identical blocks of Sylvester matrices having the same eigenvalues as $A(t, \xi)$ and $\mathcal{B}(t, D_x)$ is composed of $m \times m^2$ blocks with only the last row not identically zero. Since the original homogeneous system has been transformed into a system with lower order terms, to get well-posedness of the corresponding Cauchy problem (SylvesterCP), we need to find some Levi-type conditions. These are obtained by following the ideas for scalar equations in [1].

Step 5 We apply the partial Fourier transform with respect to x to (SylvesterCP) and we prove an energy estimate from which the assertions of the well-posedness theorems follow in a standard way. A key point is the construction of the quasi-symmetriser of the matrix $\mathcal{A}(t, \xi)$; see [2].

Statements of the well-posedness results

We assume that the matrix $\mathcal{B}(t, \xi)$ in (SylvesterCP) satisfies

$$\sum_{k=1}^m |b_{kj}^{(l)}(t, \xi)|^2 \prec \sum_{i=1}^m |\sigma_{m-1}^{(m-1)}(\pi_i \lambda)|^2 \quad (\text{Levi})$$

for any $l = 1, \dots, m-1$ and $j = 1, \dots, m$.

Theorem 1. Let $A(t, D_x)$, $t \in [0, T]$, $x \in \mathbb{R}^n$, be an $m \times m$ matrix of first order differential operators with C^∞ -coefficients. Let $A(t, \xi)$ have real eigenvalues satisfying condition (Roots). Let

$$\begin{cases} D_t u - A(t, D_x)u = 0, & (t, x) \in [0, T] \times \mathbb{R}^n \\ u|_{t=0} = u_0, & x \in \mathbb{R}^n \end{cases}$$

be the Cauchy problem (CP). Assume that the Cauchy problem (SylvesterCP), obtained from (CP) by block Sylvester reduction and the lower order terms matrix $\mathcal{B}(t, D_x)$ fulfills the Levi-type conditions (Levi). Hence, for all $s \geq 1$ and for all $u_0 \in \gamma^s(\mathbb{R}^n)^m$ there exists a unique solution $u \in C^1([0, T], \gamma^s(\mathbb{R}^n)^m)$ of the Cauchy problem (CP).

The next theorem gives a condition that is easier to check and yields the assertion of the last theorem. However, in general, the condition is more restrictive.

Theorem 2. Let $A(t, D_x)$, $t \in [0, T]$, $x \in \mathbb{R}^n$, be an $m \times m$ matrix of first order differential operators with C^∞ -coefficients. Let A have real eigenvalues satisfying condition (Roots) and let $Q = (q_{ij})$ be the symmetriser of $A_0 = \langle \xi \rangle^{-1} A$. Assume that

$$\max_{k=1, \dots, m-1} \|D_t^k A_0(t, \xi)\|^2 \prec q_{j,j}(t, \xi) \quad (\text{Levi II})$$

for all $(t, \xi) \in [0, T] \times \mathbb{R}^n$ and $j = 1, \dots, m-1$. Hence, for all $s \geq 1$ and for all $u_0 \in \gamma^s(\mathbb{R}^n)^m$ there exists a unique solution $u \in C^1([0, T], \gamma^s(\mathbb{R}^n)^m)$ of the Cauchy problem (CP).

The last theorem states C^∞ -well posedness under analyticity assumptions.

Theorem 3. If all entries of $A(t, D_x)$ in (CP) are analytic on $[0, T]$, the eigenvalues satisfy (Roots) and the entries of the matrix $\mathcal{B}(t, \xi)$ in (SylvesterCP) satisfy the Levi conditions (Levi) for ξ away from 0, then the Cauchy problem (CP) is C^∞ well-posed, i.e., for all $u_0 \in C^\infty(\mathbb{R}^n)^m$ there exists a unique solution $u \in C^1([0, T], C^\infty(\mathbb{R}^n)^m)$ of the Cauchy problem (CP).

Examples

Example 1. When $m = 2$, the Levi-type conditions (Levi) imply (Levi II). Indeed, the Levi-type conditions in that case are formulated as

$$\begin{aligned} (|D_t a_{11}(t)|^2 + |D_t a_{21}(t)|^2) \langle \xi \rangle^{-2} &\prec \lambda_1^2(t, \xi) + \lambda_2^2(t, \xi), \\ (|D_t a_{12}(t)|^2 + |D_t a_{22}(t)|^2) \langle \xi \rangle^{-2} &\prec \lambda_1^2(t, \xi) + \lambda_2^2(t, \xi). \end{aligned}$$

Example 2. In the special case $D_t^2 u - a(t) D_x^2 u = 0$ with $a(t) \geq 0$ and appropriate Cauchy data, the Levi-type condition is automatically satisfied for $a \in C^2[0, T]$. Indeed, with $a_{11} = 0$, $a_{12} = 1$, $a_{21} = a(t)$, and $a_{22} = 0$, condition (Levi II) becomes $|D_t a(t)|^2 \leq C a(t)$ which is satisfied by Glaeser's inequality.

Example 3. Let us now take a 3×3 matrix A with trace zero. For simplicity let us assume that $n = 1$ and that the eigenvalues of the corresponding A_0 are $\lambda_1(t, \xi) = -\sqrt{a(t)} \langle \xi \rangle^{-1}$, $\lambda_2(t, \xi) = 0$ and $\lambda_3(t, \xi) = \sqrt{a(t)} \langle \xi \rangle^{-1}$ with $a(t) \geq 0$ for $t \in [0, T]$. It follows that the hypothesis (Roots) on the eigenvalues is satisfied. By direct computation we get

$$\begin{aligned} q_{1,1} &= \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 = a(t) \langle \xi \rangle^{-2}, \\ q_{2,2} &= (\lambda_1 + \lambda_2)^2 + (\lambda_1 + \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 = 2a(t) \langle \xi \rangle^{-2}. \end{aligned}$$

It follows that both $q_{1,1}$ and $q_{2,2}$ are comparable to a and therefore, we conclude that

$$\begin{aligned} |b_{kj}^{(1)}|^2 &= |D_t^2 a_{kj} + 2D_t a_{kj}|^2 \prec a(t), \\ |b_{kj}^{(2)}|^2 &= |a_{k1} D_t a_{1j} + a_{k2} D_t a_{2j} + a_{k3} D_t a_{3j}|^2 \prec a(t), \end{aligned}$$

for $k = 1, 2, 3$ and $j = 1, 2$. We can easily see on the matrix

$$A(t, \xi) = \begin{pmatrix} 0 & a(t) & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xi$$

that the conditions above on the entries of A entail

$$|D_t^k a(t)|^2 \prec a(t)$$

for all $t \in [0, T]$ and $k = 1, 2$, i.e. condition (Levi II).

References

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