Summer School

Singular Integrals on Nilpotent Lie Groups and Related Topics

19.09.–23.09.2022

Information, schedule, & abstracts

Organizers:

Christian Jäh (Göttingen, Germany)
Pablo Ramacher (Marburg, Germany)
Ingo Witt (Göttingen, Germany)

Contents

1. Schedule

The introductory course on Singular Integrals on Euclidean Space and Nilpotent Groups is given by Christian Jäh (Göttingen) and the advanced courses are given by

- Véronique Fischer (Bath), *Harmonic Analysis on the Heisenberg Group and Related Topics*
- Alessio Martini (Torino), *Evolution PDEs on Nilpotent Lie Groups and Functional Calculus for Sub-Laplacians*
- Wolfram Bauer (Hanover), *Subriemannian Geometry and Nilpotent Lie Groups*
- Zhuoping Ruan (Nanjing), *Singular Integrals with Flag Kernels*

The Q & A sessions are reserved for in-depth questions and discussions on the presented material. You can also use the **Schlauch** (behind the Sitzungszimmer) to have discussions and contemplate the material. The lectures will all be in the **Sitzungszimmer**.

Time	September 19	September 20	September 21	September 22	September 23
8:30–9:00	Registration				
9:00–10:00	Jäh	Fischer	Martini	Ruan	Bauer
10:00–10:20	Coffee	Coffee	Coffee	Coffee	Coffee
10:20–11:20	Jäh	Martini	Fischer	Fischer	Ruan
11:20–12:20	Jäh	Ruan	Bauer	Bauer	Closing
12:20–14:00	Lunch	Lunch	Lunch	Lunch	
14:00–15:00	Martini	Bauer	Ruan	Fischer	
15:00–16:00	Q & A	Q & A	Q & A	Q & A	
16:00–16:20	Coffee	Coffee	Coffee	Coffee	
16:20–17:20	Bauer	Restrepo	Gómez Cobos	Martini	
17:20–18:20	Fischer	van Velhoven	Martini	Ruan	
18:20–	Social		BBQ		

2. Abstracts – Advanced Courses

2.1. **Véronique Fischer – Harmonic Analysis on the Heisenberg Group and Related Topics.**

Harmonic Analysis on the Heisenberg Group and Related Topics

Véronique Fischer
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The Heisenberg group $\mathbb{H}_n$ plays a fundamental rôle in several branches of analysis, especially in non-commutative harmonic analysis, but also in sub-Riemannian geometric analysis. Indeed, $\mathbb{H}_n$ may be viewed as the simplest example of non-commutative nilpotent Lie group and sub-Riemannian manifold. Moreover, it is related to Euclidean phase-space analysis via the Schrödinger representation.

In this series of lectures, we will discuss the following topics:

- We will start with the definition of the Heisenberg group $\mathbb{H}_n$, in particular various equivalent realisations.
- We will then introduce important objects and structures considered on the Heisenberg group coming from sub-Riemannian geometry (e.g. the canonical sub-Laplacian, the horizontal distributions, the CC-distance etc.).
- We will devote more than one lecture to harmonic analysis on the Heisenberg group in relation to representation theory and special functions.
- At the end of the lectures, we will discuss the recent progress on non-commutative phase-space analysis on the Heisenberg group and beyond.

2.2. **Wolfram Bauer– Subriemannian Geometry and Nilpotent Lie Groups. Subriemannian Geometry and Nilpotent Lie Groups**

Wolfram Bauer
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Subriemannian manifolds model systems under non-holonomic constraints and their study vastly generalizes Riemannian geometry. Global and local aspects of such spaces and their interactions have attracted strong attention during the last years. Subriemannian structures and induces PDEs nowadays appear in many different areas of mathematics and physics as well as in applied sciences. Applications arise in control theory, hypoelliptic diffusion, geometric measure theory, potential analysis, optimal transport or harmonic analysis.

In analogy to the tangent space in Riemannian geometry, nilpotent Lie groups form the local "flat" models of a Subriemannian manifold. Their study forms the first step in the analysis and local approximation of "curved" Subriemannian spaces. Generalizing the Laplace-Beltrami operator in Riemannian geometry one considers the hypoelliptic sub-Laplacian Δ_{sub} on M. This operator and the induced heat equation on M and, in particular, on nilpotent Lie groups is our central object of interest. In fact, it provides links between analytic (e.g. spectral theoretical) and geometric quantities.

During five lectures we aim to sheet some light on the role of nilpotent Lie groups G and hypoelliptic operators on G to the analysis of Subriemannian manifolds. "Sum-of-squares" of left-invariant differential operators on nilpotent Lie groups are interesting in their own right and can be applied to prove the existence of fundamental solutions of homogeneous Hörmander operators based on global lifting methods. Depending on time and interest of the participants we will discuss the following topics:

1. Introduction to Subriemannian geometry: basic properties and examples
2. Nilpotent approximation, Popp measure construction and intrinsic sub-Laplacian
3. Small time heat kernel expansions for degenerate operators
4. Folland's global lifting theorem and fundamental solution of degenerate operators
5. Local solvability for a class ultra-hyperbolic operators on pseudo-H-type groups

2.3. **Alessio Martini – Evolution PDEs on Nilpotent Lie Groups and Functional Calculus for Sub-Laplacians.**

Evolution PDEs on nilpotent Lie groups and functional calculus for sub-Laplacians

Alessio Martini
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Stratified nilpotent Lie groups G can be considered as local models of more general sub-Riemannian manifolds. As a manifold, such a group G is diffeomorphic to $\mathbb{R}^n$, but its algebraic structure (given by a polynomial group law) is in general noncommutative. On the other hand, differently from an arbitrary manifold, on a stratified group G we have natural translations and dilations, which are compatible with the algebraic structure and allow us (much as on $\mathbb{R}^n$) to relate local and global phenomena.

In this context, the natural substitute for the Laplace operator is what is known as a *homogeneous, left-invariant sub-Laplacian* on G. In terms of such a (positive-definite) sub-Laplacian $\mathcal{L}$, one may make sense of analogues of classical evolution PDEs on G, including the heat equation

$$\partial_t u + \mathcal{L}u = 0,$$

the wave equation

$$\partial_t^2 u + \mathcal{L}u = 0,$$

and the Schrödinger equation

$$\partial_t u + i\mathcal{L}u = 0.$$

The analysis of the above equations cannot always proceed as in the classical case, due among other things to the fact that the sub-Laplacian $\mathcal{L}$ is not an elliptic operator, but is just sub-elliptic. While a number of properties analogous to the classical ones can be recovered, nevertheless one also discovers new and potentially unexpected phenomena, which are specific to the sub-elliptic world and reflect the features of the underlying sub-Riemannian geometry. An especially useful companion in this analysis turns out to be the functional calculus for the sub-Laplacian $\mathcal{L}$, and in particular the study of under what conditions on a bounded function $F : [0, \infty) \to \mathbb{C}$ one can ensure that the operator $F(\mathcal{L})$, initially defined and bounded on $L^2(G)$ by the Spectral Theorem, extends to a bounded operator on $L^p(G)$ for some $p \neq 2$.

In these series of lectures we plan to touch on the following topics:

- heat kernel estimates and finite propagation speed for the wave equation;
- from heat kernel estimates to a differentiable L^p functional calculus for the sub-Laplacian;
- a Mehler type formula on 2-step groups, and (lack of) dispersive estimates for the Schrödinger equation;
- Miyachi–Peral type estimates for the wave equation and sharp L^p multiplier theorems.

2.4. **Zhuoping Ruan – Singular Integral Operators with Flag Kernels.**

Singular integral operators with flag kernels

Zhuoping Ruan
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Multi-parameter theory in Fourier analysis has a long history. It starts with the strong maximal function of B. Jessen, J. Marcinkiewicz, and A. Zygmund in 1935. The class of singular integral operators with flag kernels reflects a multi-parameter structure. This class of operators arose initially in the study of the composition of sub-elliptic operators on the Heisenberg group (such as the sub-Laplacian $\mathcal{L}$ and $\Box_b$). In particular, D. Müller, F. Ricci, and E. M. Stein in 1995 showed that the convolution kernel of a Marcinkiewicz multiplier on the Heisenberg group is a flag kernel. The notion of such kernel was formalized by A. Nagel, F. Ricci, and E. M. Stein in 2001 as a tool to study operators associated with the $\bar{\partial}$- Neumann problem on certain CR manifolds. Further applications result from sub-elliptic problems on nilpotent Lie groups.

In the five lectures, we will introduce the class of singular integral convolution operators T on nilpotent Lie groups G with flag kernels. These kernels are product-type distributions with a special structure so that their singularities are supported on a standard flag $(0) \subsetneq V_1 \subsetneq \ldots \subsetneq V_k \subsetneq G$. Flag kernels can be characterized in a number of different, but equivalent ways:

- in terms of size and cancellation conditions,
- in terms of their Fourier transforms, and
- in terms of decompositions into dyadic sums of dilates of bump functions.

We will mainly be concerned with the introduction of techniques and discussion of approaches from A. Nagel, F. Ricci, E. M. Stein and S. Wainger [1, 2]. The goal is to show that this class of operators T form an algebra under composition and that these operators T are bounded on $L^p(G)$ for $1 < p < \infty$.

The outline of these lectures is as follows:

1. Dilations and flag kernels on G
2. Fourier transform duality of flag kernels and flag multipliers
3. Strong and weak cancellations
4. Dyadic decomposition of flag kernels
5. Dyadic sums of bump functions with weak cancellation
6. Composition of convolution operators with flag kernels
7. Maximal functions
8. Boundedness of flag convolution T on $L^p(G)$ for $1 < p < \infty$

References

[1] A. Nagel, F. Ricci, E. M. Stein, and S. Wainger, Singular integrals with flag kernels on homogeneous groups, I. Rev. Mat. Iberoam. 28 (2012), 631-722.

[2] ———, Algebras of singular integral operators with kernels controlled by multiple norms. Mem. Amer. Math. Soc. 256 (2018), vii+141pp.

3. Abstracts – Short Communications

3.1. Tuesday I.

Heat and wave type equations with fully non-local operators on locally compact groups

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We study heat and wave type equations on a locally compact separable unimodular group G by using a non-local differential operator in time and a positive left invariant operator (maybe unbounded) on G. For the considered equations, we prove the $L^p(G) - L^q(G)$ norm estimates of the solutions with $1 < p \leqslant 2 \leqslant q < +\infty$. We also provide some asymptotic estimates (large-time behaviour) for the solutions. Some examples on graded, Heisenberg and compact Lie groups are given.

3.2. Tuesday II.

Smooth frames in the orbit of a square-integrable representation of a nilpotent Lie group

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I will present various results on the existence of frames in the orbit of a square-integrable representation of a nilpotent Lie group. The key theorems concern sharp sufficient and necessary conditions for the existence of smooth vectors generating such a frame. These conditions are formulated in terms of the Beurling density of the index set and the formal degree of the representation, whose ratio quantifies the redundancy of a frame.

3.3. Wednesday I.

Global Pseudodifferential Calculus for Manifolds

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Pseudodifferential operators on smooth manifolds are defined classically by means of local coordinates using the well-known theory for $\mathbb{R}^n$. Since we would like to have invariance under change of coordinates this will naturally imply some restrictions on the classes that we can construct, particularly we are faced with the restriction $\rho > 1/2$ on the classes $S_{(,)}^m$. Safarov introduced a pseudodifferential calculus on smooth manifolds without using local coordinates, instead he uses a linear connection ∇ to obtain a global calculus. Using this construction and assuming that the connection is symmetric he was able to weaken the aforementioned restriction to $\rho > 1/3$. I will introduce the Safarov pseudodifferential calculus and and I will present some progress on the $L^p - L^p$ estimates for this calculus.

4. Where to go for lunch?

For the Zentralmensa, you need a guest card to pay as there is no cash payment at the moment. You can get the guest card at the information desk at the mensa (on the first day, someone of us will help you) and you can then charge it on a machine with an EC card.

- Zentralmensa (Google maps). This is the main mesa on the central campus. This is about 20 minutes walk from the institute.[1] The menu can be found here (choose Zentralmensa in the drop-down). The menu is also displayed in English at the mensa.
  In the Café Zentral a selection of hot and cold sandwiches and pizzas as well as sweets is offered.
- Bakery Küster (Google Maps).
  They have a great selection of hot and cold sandwiches, salads, and other snacks. Not to forget sweets.
  Here is a link to the Cafe menu. (German)
- Bakery Holzofenbäckerei (Google maps).
  Here you can get sandwiches, soup, and some other snacks.

[1]Can be connected with a stroll to Gauß' grave on the way back.

5. Where to go for dinner?

6. The University and the city of Göttingen

Some sights/museums:

- Forum Wissen – the new museum of knowledge in Göttingen – aims to make research with objects tangible., (Admission free.)
- Gänselisel on the main market place in front of the old city hall.
  PhD student traditionally bring her, following a successful defence, flowers and give her a kiss.
- Gauß' grave
- Gauß Observatory
- City Cemetery with graves of David Hilbert, Max Born, Otto Hahn, Max Planck and many more.

7. RTG 2491 - Fourier Analysis and Spectral Theory

The Research Training Group (RTG) Fourier Analysis and Spectral Theory is a joint research and graduate education programme funded by the German Science Foundation (DFG). It is based at the Mathematical Institute, University of Göttingen, with participating researchers from the University of Hanover. The initiative started October 1st, 2019 with the opening of ten PhD positions and one postdoctoral position.

The RTG Fourier Analysis and Spectral Theory is taking an interdisciplinary and innovative approach to the classical and powerful machinery modern harmonic and Fourier analysis and spectral theory. We focus on its development in the context of mathematical physics, topology and analytic number theory.

A core theme of the RTG is analysis and spectral geometry on Riemannian manifolds, in particular, locally symmetric spaces or more generally spaces acted on by groups. Besides a topological structure, in many interesting cases they also have some arithmetic or combinatorial structure, and one of the key questions involves the fascinating interplay between the spectral properties of certain associated operators on the one hand, and geometric, topological or arithmetic properties on the other. Some prototypical examples of this interaction featured in this RTG are the spectral theory of Cayley graphs of groups; analytic L^2-invariants, which link harmonic analysis to topology; and the resolvent and scattering theory of geometric differential operators on singular manifolds. A cornerstone at the interface of modern analytic number theory and harmonic analysis is the theory of automorphic forms, viewed as eigenfunctions of a family of operators on a locally symmetric space. Fourier and harmonic analysis also appear prominently in many applications of classical analytic number theory, in the representation theory of Lie groups and groupoids, and in the construction of quantum field theories with microlocal methods.

On the methodological side we, draw from a variety of analytic techniques, such as microlocal analysis, symbolic calculus, trace formulas and Plancherel theory, Fourier analysis in numerous variations, spectral and scattering theory of operators, but also classical analysis such as a careful analysis of oscillatory integrals.