Evolution PDEs on nilpotent Lie groups and functional calculus for sub-Laplacians

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Stratified nilpotent Lie groups G can be considered as local models of more general sub-Riemannian manifolds. As a manifold, such a group G is diffeomorphic to $\mathbb{R}^n$, but its algebraic structure (given by a polynomial group law) is in general noncommutative. On the other hand, differently from an arbitrary manifold, on a stratified group G we have natural translations and dilations, which are compatible with the algebraic structure and allow us (much as on $\mathbb{R}^n$) to relate local and global phenomena.

In this context, the natural substitute for the Laplace operator is what is known as a *homogeneous, left-invariant sub-Laplacian* on G. In terms of such a (positive-definite) sub-Laplacian $\mathcal{L}$, one may make sense of analogues of classical evolution PDEs on G, including the heat equation

$$\partial_t u + \mathcal{L} u = 0,$$

the wave equation

$$\partial_t^2 u + \mathcal{L} u = 0,$$

and the Schrödinger equation

$$\partial_t u + i\mathcal{L} u = 0.$$

The analysis of the above equations cannot always proceed as in the classical case, due among other things to the fact that the sub-Laplacian $\mathcal{L}$ is not an elliptic operator, but is just sub-elliptic. While a number of properties analogous to the classical ones can be recovered, nevertheless one also discovers new and potentially unexpected phenomena, which are specific to the sub-elliptic world and reflect the features of the underlying sub-Riemannian geometry. An especially useful companion in this analysis turns out to be the functional calculus for the sub-Laplacian $\mathcal{L}$, and in particular the study of under what conditions on a bounded function $F : [0,\infty) \to \mathbb{C}$ one can ensure that the operator $F(\mathcal{L})$, initially defined and bounded on $L^2(G)$ by the Spectral Theorem, extends to a bounded operator on $L^p(G)$ for some $p \neq 2$.

In these series of lectures we plan to touch on the following topics:

- heat kernel estimates and finite propagation speed for the wave equation;
- from heat kernel estimates to a differentiable L^p functional calculus for the sub-Laplacian;
- a Mehler type formula on 2-step groups, and (lack of) dispersive estimates for the Schrödinger equation;
- Miyachi–Peral type estimates for the wave equation and sharp L^p multiplier theorems.