Singular integral operators with flag kernels

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Multi-parameter theory in Fourier analysis has a long history. It starts with the strong maximal function of B. Jessen, J. Marcinkiewicz, and A. Zygmund in 1935. The class of singular integral operators with flag kernels reflects a multi-parameter structure. This class of operators arose initially in the study of the composition of sub-elliptic operators on the Heisenberg group (such as the sub-Laplacian $\mathcal{L}$ and $\Box_b$). In particular, D. Müller, F. Ricci, and E. M. Stein in 1995 showed that the convolution kernel of a Marcinkiewicz multiplier on the Heisenberg group is a flag kernel. The notion of such kernel was formalized by A. Nagel, F. Ricci, and E. M. Stein in 2001 as a tool to study operators associated with the $\bar{\partial}$-Neumann problem on certain CR manifolds. Further applications result from sub-elliptic problems on nilpotent Lie groups.

In the five lectures, we will introduce the class of singular integral convolution operators T on nilpotent Lie groups G with flag kernels. These kernels are product-type distributions with a special structure so that their singularities are supported on a standard flag $(0) \subsetneq V_1 \subsetneq \ldots \subsetneq V_k \subsetneq G$. Flag kernels can be characterized in a number of different, but equivalent ways:

- in terms of size and cancellation conditions,
- in terms of their Fourier transforms, and
- in terms of decompositions into dyadic sums of dilates of bump functions.

We will be mainly concerned with the introduction of techniques and discussion of approaches from A. Nagel, F. Ricci, E. M. Stein and S. Wainger [1, 2]. The goal is to show that this class of operators T form an algebra under composition and that these operators T are bounded on $L^p(G)$ for $1 < p < \infty$.

The outline of these lectures is as follows:

1. Dilations and flag kernels on G
2. Fourier transform duality of flag kernels and flag multipliers
3. Strong and weak cancellations
4. Dyadic decomposition of flag kernels
5. Dyadic sums of bump functions with weak cancellation
6. Composition of convolution operators with flag kernels

7. Maximal functions
8. Boundedness of flag convolution T on $L^p(G)$ for $1 < p < \infty$

References

[1] A. Nagel, F. Ricci, E. M. Stein, and S. Wainger, Singular integrals with flag kernels on homogeneous groups, I. Rev. Mat. Iberoam. 28 (2012), 631-722.

[2] ———, Algebras of singular integral operators with kernels controlled by multiple norms. Mem. Amer. Math. Soc. 256 (2018), vii+141pp.