

Evolution PDEs on Nilpotent Lie Groups and Functional Calculus For Sub-Laplacians

Alessio Martini
(Politecnico di Torino)

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LECTURE 1

Main References

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- W.-L. Chow, Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung, Math. Ann. 117 (1939), 98-105.
 - E. Nelson & W.F. Stinespring, Representation of elliptic operators in an enveloping algebra, Amer. J. Math. 81 (1959), 547-560.
 - L. Hörmander, Hypoelliptic second-order differential equations, Acta Math. 119 (1967), 147-171.

G stratified Lie group: connected, simply connected nilpotent Lie group whose Lie algebra is stratified

1) stratified Lie algebra: $(s\text{-step})$ $\underline{\mathfrak{g}} = \bigoplus_{j=1}^s \underline{\mathfrak{g}}_j$ $\underline{\mathfrak{g}}_j$ j th layer (subspace of $\underline{\mathfrak{g}}$)
 $[\underline{\mathfrak{g}}_i, \underline{\mathfrak{g}}_j] = \underline{\mathfrak{g}}_{i+j}$ ($\underline{\mathfrak{g}}_j = 0$ for $j > s$)
 $\underline{\mathfrak{g}}_1$ generates $\underline{\mathfrak{g}}$

2) $\underline{\mathfrak{g}}$ nilpotent $\Rightarrow \exp: \underline{\mathfrak{g}} \rightarrow G$ diffeo \leadsto we can identify G with $\underline{\mathfrak{g}}$ as a manifold
 G simply conn. (exponential coordinates)

3) product law is polynomial: truncation of the Baker-Campbell-Hausdorff formula
 $x \cdot y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] - [y, [x, y]]) + \dots$
 and $\boxed{x^{-1} = -x}$, $\boxed{e = 0}$.

4) left & right translations have "unipotent" differential (BCH at the linear part of BCH)
 \leadsto the Lebesgue measure on $\underline{\mathfrak{g}}$ is the (left & right) Haar measure on G

5) automorphic dilations: if $x = (x_1, \dots, x_s) \in \underline{\mathfrak{g}}_1 \oplus \dots \oplus \underline{\mathfrak{g}}_s = \underline{\mathfrak{g}} \simeq G$, we write
 $\delta_\lambda(x_1, \dots, x_s) = (\lambda x_1, \lambda^2 x_2, \dots, \lambda^s x_s) \quad \forall \lambda > 0$; then $\delta_\lambda: \underline{\mathfrak{g}} \rightarrow \underline{\mathfrak{g}}$ automorphism ($\delta_\lambda[x, y] = [\delta_\lambda x, \delta_\lambda y]$)
 $G \rightarrow G \quad \text{---} \quad (\delta_\lambda(x \cdot y) = (\delta_\lambda x)(\delta_\lambda y))$

6) Convolution on G : $f * g(x) = \int_G f(xy^{-1})g(y)dy$ (int. wrt Haar meas).

(at least for "nice" functions; also extends to distributions etc.)

• Young's convolution inequality: $\|f * g\|_r \leq \|f\|_p \|g\|_q$ $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $p, q, r \in [1, \infty]$.

• Schwartz kernel theorem: any $T: S(G) \rightarrow S'(G)$ left-invariant op.
(i.e. commutes w/ left translations)
 \downarrow
Schwartz class ($= S(\underline{g})$)

is a (right) convolution operator: $T\phi = \phi * k_T$, $k_T \in S'(G)$ conv. kernel of T .

7) left-invariant vector fields: $\underline{g} = \text{Lie}(G)$ is the Lie algebra of l.i. vector fields on G
(& diff. φ 's) (where the Lie bracket goes to "commutator" of vector fields)

as the group operation is polynomial, one can see that l.i. vector fields are
v. fields on \underline{g} w/ polynomial coefficients

Fix a basis $\underline{X}_1, \dots, \underline{X}_r$ of \underline{g}_1 . For $I = (i_1, \dots, i_N) \in \{1, \dots, r\}^N$ (non-commutative multi-index)

we write $X^I = X_{i_1} \dots X_{i_N}$. Then (by Poincaré-Birkhoff-Witt Thm)

any l.i. diff. op on G is a linear combination of X^I .

8) sub-Laplacian: $\mathcal{L} = -\sum_{j=1}^r X_j^2$ homogeneous, 2nd order left-invariant diff op on G
 \downarrow
 $\mathcal{L}(f \circ \delta_\lambda) = \lambda^2 (\mathcal{L} f) \circ \delta_\lambda$

\leadsto inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ on \mathfrak{g}_1 that makes X_1, \dots, X_r into an orthonormal basis

[any other basis \tilde{X}_j of \mathfrak{g}_1 for which $\tilde{\mathcal{L}} = -\sum_j \tilde{X}_j^2$ induces the SAME inner product]

9) sub-Riemannian metric structure:

• by left translations, \mathfrak{g}_1 determines a (left-invariant) "distribution", i.e. a sub-bundle $HG \subseteq TG$ where $H_x G = \text{span}\{X_j|_x : j=1, \dots, r\} \subseteq T_x G$.

• Similarly, $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ induces a (left-invariant) fibrewise inner product on HG .

HG is the HORIZONTAL DISTRIBUTION, its elements are the horizontal tangent vectors

• a horizontal curve is a curve $\gamma: [a, b] \rightarrow M$ such that $\gamma'(t) \in H_{\gamma(t)} M$ for a.a. $t \in [a, b]$ (abs. cont.)

• the length of a horizontal curve γ is $L(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle_{\mathcal{L}}^{1/2} dt$.

• the sub-Riemannian (Carnot-Carathéodory) distance on G is

$$d(x, y) = \inf \{L(\gamma) : \gamma \text{ horizontal curve joining } x \text{ to } y\}.$$

10) Chow's Theorem : any pair of pts $x, y \in G$ is joined by a horizontal curve.

Moreover (Ball-Box thm.) d induces the Euclidean topology on G .

other properties of the distance: a) left-invariance : $d(x, y) = |y^{-1}x|_Z$, $|x|_Z = d(x, 0)$
 b) homogeneity : $| \delta_\lambda(x) |_Z = \lambda |x|_Z$. HOMOGENEOUS NORM

\leadsto one can see that, if $x = (x_1, \dots, x_s) \in \underline{\mathfrak{g}}_1 \oplus \dots \oplus \underline{\mathfrak{g}}_s$, then $|x|_Z \approx |x_1| + |x_2|^{1/2} + \dots + |x_s|^{1/s}$
 \leadsto NOT (bi-Lipschitz) equivalent to Euclidean metric

$|B_r(x)| = r^Q |B_1(0)|$, $Q = \sum_j j \dim \underline{\mathfrak{g}}_j$ homogeneous dimension ($\det \delta_\lambda = \lambda^Q$)

$\Rightarrow G$ with Haar measure and sub-Riemannian distance is a DOUBLING metric-measure space

11) Hörmander's Theorem : the sublaplacian \mathcal{L} is hypoelliptic, i.e.

if $u \in \mathcal{D}'(G)$, $\Omega \subseteq G$ open, $(\mathcal{L}u)|_\Omega \in C^\infty(\Omega) \Rightarrow u|_\Omega \in C^\infty$.

[Holds more generally for op's $Y_1^2 + \dots + Y_s^2 + Y_0 + f$, where Y_0, \dots, Y_s are smooth real v.f.'s on a manifold satisfying bracket-free condition, and f is a smooth real fctn]

12) (variation 4) Nelson & Stinespring's Theorem : \mathcal{L} is ess. self-adj. on $L^2(G)$

13) FUNCTIONAL CALCULUS for the sub-laplacian

sp. thm: $\mathcal{L} = \int_0^\infty \lambda dE_{\mathcal{L}}(\lambda)$, $F(\mathcal{L}) = \int_0^\infty F(\lambda) dE_{\mathcal{L}}(\lambda)$ Borel fctnal calculus, $F: [\lambda_1, \infty) \rightarrow \mathbb{C}$ Borel.

\mathcal{L} left-invariant \rightarrow any $F(\mathcal{L})$ is too, so $F(\mathcal{L})\phi = \phi * k_{F(\mathcal{L})}$, $k_{F(\mathcal{L})} \in S'(G)$ (if F is bdd)

HOMOGENEITY: $\mathcal{L}(f \circ \delta_t) = t^2 (\mathcal{L}f) \circ \delta_t \rightsquigarrow F(\mathcal{L})(f \circ \delta_t) = (F(t^2 \mathcal{L})f) \circ \delta_t$

so: $k_{F(\mathcal{L})} = t^Q k_{F(t^2 \mathcal{L})} \circ \delta_t$.

$$\begin{array}{ccc} (f \circ \delta_t) * k_{F(\mathcal{L})} & & (f * k_{F(t^2 \mathcal{L})}) \circ \delta_t \\ & \uparrow & \\ (f \circ \delta_t) * (t^Q k_{F(t^2 \mathcal{L})} \circ \delta_t) & & \end{array}$$

14) Some evolution eq's:

HEAT eq. $\begin{cases} \partial_t u = -\mathcal{L}u \\ u|_{t=0} = f \end{cases} \rightarrow u = e^{-t\mathcal{L}} f = f * \underbrace{k_{e^{-t\mathcal{L}}}}_{\text{heat kernel}}$

SCHRÖDINGER eq. $\begin{cases} \partial_t u = i\mathcal{L}u \\ u|_{t=0} = f \end{cases} \rightsquigarrow u = e^{it\mathcal{L}} f$

WAVE eq. $\begin{cases} \partial_t^2 u = -\mathcal{L}u \\ u|_{t=0} = f \\ \partial_t u|_{t=0} = g \end{cases} \rightsquigarrow u = \cos(t\sqrt{\mathcal{L}})f + \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}g$
 $(= e^{it\sqrt{\mathcal{L}}} a + e^{-it\sqrt{\mathcal{L}}} b)$