Evolution PDEs on Nilpotent Lie Groups and Functional Calculus for Sub-Laplacians

Alessio Martini
(Politecnico di Torino)

summer school "Singular Integrals on Nilpotent Lie Groups and Related Topics"
Göttingen, 19.-23.9.2022

LECTURE 2

Main References

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- A. Sikora, Riesz transforms, Gaussian bounds and the method of wave equation, Math. Z. 247 (2004), 643-662.
- A. Hulanicki & J.W. Jenkins, Almost everywhere summability on nilmanifolds, TAMS 278 (1983), 703-715.

- A. Martini, Spectral theory for commutative algebras of differential operators on Lie groups, JFA 260 (2011), 2767-2814.
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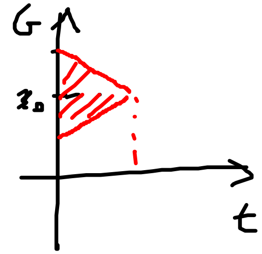
Heat & Wave Propagators: basic facts (rel. w/ C-C distance)

1) FINITE PROPAGATION SPEED: if $u \in C^2([0,T]\times G)$ is sol. of wave eq. $\partial_t^2 u + \mathcal{L} u$ with $u(x,0) = \partial_t u(x,0) = 0$ for $x \in \overline{B}_r(x_0)$

[Melrose '86]

then $u \equiv 0$ on the conical region $\Gamma = \{(t,x) \in [0,T] : d(x,x_0) \le r - t\}$



⤳ rephrased in terms of wave propagator:

$$\boxed{\mathrm{supp}\left(K_{\cos(t\sqrt{\mathcal{L}})}\right) \subseteq \overline{B}(0,r)}$$

2) (Gaussian type) HEAT KERNEL ESTIMATES: $|X^I k_{e^{-t\mathcal{L}}}(x)| \lesssim_I t^{-\frac{Q+|I|}{2}} e^{-c|x|_{\mathcal{L}}^2/t}$

[Varopoulos '85-'86 ?]

note: by homogeneity: $k_{F(t^2\mathcal{L})} = t^{-Q} k_{F(\mathcal{L})} \circ \delta_{t^{-1}}$, $k_{e^{-t\mathcal{L}}} = t^{-Q/2} k_{e^{-\mathcal{L}}} \circ \delta_{t^{-1/2}}$

Hunt's Theorem: the $k_{e^{-t\mathcal{L}}}$ are probability measures on G

by hypoellipticity (of $\partial_t + \mathcal{L}$ on $\mathbb{R}\times G$) and homogeneity: $k_{e^{-t\mathcal{L}}} \in \mathcal{S}(G)$, $k_{e^{-t\mathcal{L}}} > 0$, $\|k_{e^{-t\mathcal{L}}}\|_1 = 1$

by positivity: $\|k_{e^{-t\mathcal{L}}}\|_\infty = k_{e^{-t\mathcal{L}}}(0) = t^{-Q/2} k_{e^{-\mathcal{L}}}(0)$ ⤳ we have on-diagonal estimate

This, combined w/ FPS (see, e.g., Sikora '04) leads to Gaussian-type bounds

[Other relevant ref. is book by Varopoulos–Saloff-Coste–Coulhon '92]

Proof of FPS: [Following D. Müller's Padova 2004 lecture notes]

basic facts: • $d(\cdot,y)$ is 1-Lipschitz wrt d $\quad\forall y\in G$

• Let $\varphi\in C_c^\infty(G)$, $\varphi>0$, $\int\varphi=1$ and define approx. id. $\varphi_\varepsilon(x)=\varepsilon^{-Q}\varphi(\delta_{\varepsilon^{-1}}(x))$

Let $d_\varepsilon(\cdot,y)=\varphi_\varepsilon * d(\cdot,y)$. Then • $d_\varepsilon(\cdot,y)\to d(\cdot,y)$ uniformly as $\varepsilon\to 0$,

• $d_\varepsilon(\cdot,y)$ is 1-Lipschitz wrt d

• $d_\varepsilon\in C^\infty$

• Let $X=\sum_{j=1}^r \xi_j X_j$, $\quad\xi_j\in\mathbb{R}^r$, $\sum_j \xi_j^2=1$ (X is unit horizontal vector)

Then for each $x\in G$, the flow curve of X $\quad t\mapsto x\exp(tX)$ starting from x is a horizontal curve w/ unit tangent vector

$$\Rightarrow \quad |d_\varepsilon(x\exp(tX),y)-d_\varepsilon(x,y)|\le|t| \qquad \forall t$$

and by differentiation in t we obtain $\boxed{|X d_\varepsilon(\cdot,y)|\le 1}$ $\quad\forall y\in G$, $\forall X$ unit horiz. vector

$$\Rightarrow \quad \underbrace{\Big(\sum_j |X_j d_\varepsilon(\cdot,y)|^2\Big)^{1/2}=\sup_{|\xi|\le 1}\Big|\sum_j \xi_j X_j d_\varepsilon(\cdot,y)\Big|\le 1.}_{\text{referred to as Lemma below}}$$

Energy estimate: Assume $u \in C^2([0,T]\times G)$ solves wave eq. $\partial_t^2 u + \mathcal{L}u = 0$

and $u(0,x) = \partial_t u(0,x) = 0$ for $x \in \overline{B}_r(x_0)$.

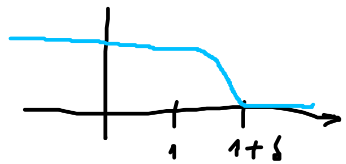
- We want to show: $u \equiv 0$ on $T = \{(t,x) : d(x,x_0) \le r-t\}$ | WLOG assume u REAL.

  i.e. $u(x,t) = 0$ on $\overline{B}_{r-t}(x_0)$ for $0 \le t \le r$.

- Energy $E(t) = \frac{1}{2}\int_{B_{r-t}(x_0)} \left[(\partial_t u(t,x))^2 + \sum_j (X_j u(t,x))^2\right] dx$ → ENOUGH to show: $E(t) = 0 \ \forall t \le r$

Smooth approx ($\delta > 0$): Choose $\chi = \chi_\delta \in C^\infty(\mathbb{R})$, $\chi \equiv 1$ on $(-\infty, 1]$, $\chi \equiv 0$ on $[1+\delta, \infty)$

$\chi' \le 0$



Define $\tilde{E}(t) = \tilde{E}_{\varepsilon,\delta}(t) = \frac{1}{2}\int_G \left[(\partial_t u(t,x))^2 + \sum_j (X_j u(t,x))^2\right] \chi_\delta\left(\frac{d_\varepsilon(x,x_0)}{r-t}\right) dx$

Differentiate:
$$\partial_t \tilde{E}(t) = \int_G \left[(\partial_t u)(\partial_t^2 u) + \sum_j (X_j u)(X_j \partial_t u)\right] \chi_\delta\left(\frac{d_\varepsilon}{r-t}\right)$$
$$+ \frac{1}{2}\int_G \left[(\partial_t u)^2 + \sum_j (X_j u)^2\right] \chi_\delta'\left(\frac{d_\varepsilon}{r-t}\right) \frac{d_\varepsilon}{(r-t)^2} =: J_1 + J_2$$

Now: $\sum_j X_j\big((X_j u)(\partial_t u)\big) \underset{\text{Leibniz}}{=} \sum_j \left[(X_j^2 u)(\partial_t u) + (X_j u)(X_j \partial_t u)\right] = -(\mathcal{L}u)\,\partial_t u + \sum_j (X_j u)(X_j \partial_t u) \underset{\text{PDE}}{=} (\partial_t^2 u)(\partial_t u) + \sum_j (X_j u)(X_j \partial_t u)$

$$\Rightarrow J_1 = \sum_j \int_G X_j\big((X_j u)\,\partial_t u\big)\, \chi_\delta\left(\frac{d_\varepsilon}{r-t}\right) \overset{\text{by parts}}{=} -\sum_j \int_G (X_j u)(\partial_t u)\, \chi_\delta'\left(\frac{d_\varepsilon}{r-t}\right) \frac{X_j d_\varepsilon}{r-t}$$

$$J_1 = -\sum_j \int_G (X_j u)(\partial_t u)\, \chi_\delta'\left(\frac{d_\varepsilon}{r-t}\right) \frac{X_j d_\varepsilon}{r-t},$$

$$|J_1| \underset{CS}{\leq} \int_G |\partial_t u| \sqrt{\sum_j (X_j u)^2}\, \underbrace{\sqrt{\sum_j (X_j d_\varepsilon)^2}}_{\substack{\leq 1 \\ \text{(Lemma)}}}\, \left|\chi'\left(\frac{d_\varepsilon}{r-t}\right)\right| \frac{1}{r-t}$$

$$\underset{\chi' \leq 0}{\leq} -\int \frac{1}{2} \int \left[(\partial_t u)^2 + \sum_j (X_j u)^2\right] \chi'\left(\frac{d_\varepsilon}{r-t}\right) \overbrace{\frac{d_\varepsilon}{r-t}}^{\geq 1 \text{ on supp } \chi} \frac{1}{r-t} = -J_2 \quad (!)$$

So $\tilde{E}_{\varepsilon,\delta}'(t) = J_1 + J_2 \leq |J_1| + J_2 \leq 0 \implies t \mapsto \tilde{E}_{\varepsilon,\delta}(t)$ decreasing

$\implies$ $t \mapsto E(t)$ decreasing

(by taking $\varepsilon \to 0$ and then $\delta \to 0$)

BUT $E(0) = 0$ by assumption $\implies E(t) = 0 \quad \forall t \leq r$

$\implies (\partial_t u)^2 + \sum_j (X_j u)^2 \equiv 0$ on T $\implies$ u constant on T (here we use bracket-gen-condition !)

$\implies u \equiv 0$ on T (& Chow's thm to find horiz. curves joining pts of section of T)

Some consequences on Functional Calculus:

(1) From FPS: if $F:\mathbb{R}\to\mathbb{C}$ is even and $\operatorname{supp}\hat{F}\subseteq[-r,r]$ $(r>0)$

then $\operatorname{supp} k_{F(\sqrt{\mathcal{L}})}\subseteq\overline{B}(0,r)$.

Indeed $F(\mathcal{L})=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{F}(\tau)\cos(\tau\sqrt{\mathcal{L}})$

(2) From heat kernel bounds: Let $F:[0,\infty)\to\mathbb{C}$ Borel be bdd & comp.supp.; then

$F(\mathcal{L})=F(\mathcal{L})e^{\mathcal{L}}e^{-\mathcal{L}}=\tilde{F}(\mathcal{L})e^{-\mathcal{L}}$ where $\tilde{F}(\lambda)=F(\lambda)e^{\lambda}$ is also bdd & cpt.supp.

$\Longrightarrow$ $k_{F(\mathcal{L})}=k_{\tilde{F}(\mathcal{L})e^{-\mathcal{L}}}=k_{\tilde{F}(\mathcal{L})}*k_{e^{-\mathcal{L}}}=\tilde{F}(\mathcal{L})k_{e^{-\mathcal{L}}}\in L^2(G)$ as $\tilde{F}(\mathcal{L})$ is L^2-bdd.

and similarly $X^I k_{F(\mathcal{L})}=k_{\tilde{F}(\mathcal{L})}*X^I k_{e^{-\mathcal{L}}}\in L^2(G)*(L^1\cap L^2(G))\subseteq L^2\cap L^\infty(G)$ $\forall I$ multiindex

$\Longrightarrow$ $k_{F(\mathcal{L})}$ is smooth w/ all l.i. derivatives in $L^2(G)\cap L^\infty(G)$

$\rightsquigarrow$ we can define a Plancherel measure (regular Borel measure) $\sigma_{\mathcal{L}}$ on $[0,\infty)$ by

$\|k_{F(\mathcal{L})}\|_2^2=\int_{[0,\infty)}|F(\lambda)|^2\,d\sigma_{\mathcal{L}}(\lambda)$ $\rightsquigarrow$ by homogeneity: $d\sigma(\lambda)=\lambda^{Q/2}\frac{d\lambda}{\lambda}$ on $(0,\infty)$

[cf. Hulanicki & Jenkins '83, Christ '91]