Evolution PDEs on Nilpotent Lie Groups and Functional Calculus for Sub-Laplacians

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LECTURE 3

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$\mathcal{L}$ sub-Laplacian on stratified group G

<u>Finite prop. speed</u>: $\operatorname{supp} k_{F(\sqrt{\mathcal{L}})} \subseteq \bar{B}(0,r)$ if F even, $\operatorname{supp} \hat{F} \subseteq [-r,r]$

<u>Plancherel formula</u>: $\| k_{F(\mathcal{L})} \|_2^2 = c \int_0^\infty |F(\lambda)|^2 \lambda^{Q/2} \frac{d\lambda}{\lambda}$

<u>Hulanicki's Theorem</u>: If $F \in \mathcal{S}(\mathbb{R})$ then $k_{F(\mathcal{L})} \in \mathcal{S}(G)$.

<u>Christ</u> / <u>Mauceri & Meda</u>: 1) If $F \in L^2_s(\mathbb{R})$ (Sobolev space), $s > \frac{Q}{2}$, and $\operatorname{supp} F \subseteq [1/2, 2]$

then $k_{F(\mathcal{L})} \in L^1(G)$, $\| k_{F(\mathcal{L})} \|_1 \lesssim_K \| F \|_{L^2_s}$.

(by homogeneity: $\sup_{t>0} \| k_{F(t\mathcal{L})} \|_1 \lesssim \| F \|_{L^2_s}$)

2) if $\sup_{t>0} \| F(t\cdot)\chi \|_{L^2_s} < \infty$ for some $s > Q/2$ and $0 \not\equiv \chi \in C_c^\infty((0,\infty))$, then

$F(\mathcal{L})$ is w.t. (1,1) and L^p-bdd for $1<p<\infty$. (Mihlin-Hörmander-type theorem).

Proof sketch of <u>Christ</u> / <u>Mauceri-Meda thm</u>:

- 2) Follows from 1) via "standard" Calderón-Zygmund theory [cf. Duong & McIntosh '99]
- 1) Follows by C-S from <u>weighted L^2-estimate</u>: $\| (1+|\cdot|_{\mathcal{L}})^\alpha k_{F(\mathcal{L})} \|_2 \lesssim_{\alpha,\beta} \| F \|_{L^2_\beta}$, $\beta > \alpha$ $(*)$

  (cf. case of $\mathbb{R}^n$: $k_{F(\mathcal{L})} = \mathcal{F}^{-1} F(|\cdot|^2)$ and $(*)$ is trivial (w/ $\beta = \alpha$) by char. of L^2_s-norm via F.T.)

Proof of the weighted L^2-estimate:

1. reduction to functional calculus of $M = e^{-t\mathcal{L}}$:

$F(\mathcal{L}) = G(M)$ where $G(\lambda) = F(-\log \lambda)$; if $\operatorname{supp} F \subseteq [1/2, 2]$, then $\operatorname{supp} G \subseteq [e^{-2}, e^{-1/2}] \subseteq (-\pi,\pi)$

2. Fourier series decomposition:

$$G(\lambda) = \sum_{n\in\mathbb{Z}} \hat{G}(k) e^{ik\lambda} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{G}(k)(e^{ik\lambda}-1) \implies F(\mathcal{L}) = G(M) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{G}(k)(e^{ikM}-1)$$

$\lambda \in (-\pi,\pi)$ $\quad$ $G(0)=0$

IDEA: if $\|e^{ikM}-1\| \lesssim |k|^\alpha$ then $\|G(M)\| \lesssim \sum_k |\hat{G}(k)|\,|k|^\alpha \rightsquigarrow$ smoothness condition on G

Fundamental tool: weighted Young's inequality:

let $\|f\|_{p,\alpha} = \|(1+|\cdot|_{\mathcal{Z}})^\alpha f\|_p$, $\|f\|_{p,\exp} = \|\exp(|\cdot|_{\mathcal{Z}}) f\|_p$. Then:

($\alpha \geq 0$, $p\in[1,\infty]$)

$$\|f * g\|_{p,\alpha} \leq \|f\|_{p,\alpha} \|g\|_{1,\alpha}, \quad \|f*g\|_{p,\exp} \leq \|f\|_{p,\exp} \|g\|_{1,\exp}$$

[due to subadditivity (tr. ineq. + invariance) $|xy|_{\mathcal{Z}} \leq |x|_{\mathcal{Z}} + |y|_{\mathcal{Z}}$, which implies submultiplicativity of the weights $(1+|\cdot|_{\mathcal{Z}})^\alpha$, $\exp(|\cdot|_{\mathcal{Z}})$]

Step 1: From Gaussian-type heat kernel bds it follows: $\|k_M\|_{p,\exp} < \infty \quad \forall p \in [1,\infty]$

Step 2: $e^{ik\lambda} - 1 = \sum_{n>0} \frac{(ik\lambda)^n}{n!} \Rightarrow \|k_{e^{ikM}-1}\|_{2,\exp} \leq \sum_{n>0} \frac{|k|^n \|(k_M)^{*n}\|_{2,\exp}}{n!}$ (tr. ineq.)

$$\underset{(\text{w. Young})}{\leq} \sum_{n>0} \frac{|k|^n \|k_M\|_{2,\exp} \|k_M\|_{1,\exp}^{n-1}}{n!} \lesssim e^{|k| \, \|k_M\|_{1,\exp}}$$

Step 3: (exploiting $\underline{\text{cancellation on } L^2}$): $\|k_{e^{ikM}-1}\|_2 = \left\| \frac{e^{ikM}-1}{M} k_M \right\|_2 \underset{(\text{sp. thm.})}{\leq} \|k_M\|_2 \sup_{\lambda>0} \left| \frac{e^{ik\lambda}-1}{\lambda} \right| \lesssim |k|$ (!)

Step 4 ("interpolation"): $\|k_{e^{ikM}-1}\|_{2,\alpha}^2 = \int_G |k_{e^{ikM}-1}(x)|^2 (1+|x|)_{\mathcal{I}}^{2\alpha}\, dx = \int_{|x|_{\mathcal{I}} \leq R} + \int_{|x|_{\mathcal{I}} > R}$ (any $R>0$)

(Let $\alpha \geq 0$)

$$\lesssim_\alpha (1+R)^{2\alpha} |k|^2 + \left[\sup_{\lambda > R} (1+\lambda)^{2\alpha} e^{-2\lambda} \right] e^{2c|k|}$$

$\underline{\Rightarrow}$ $\|k_{e^{ikM}-1}\|_{2,\alpha} \lesssim (1+R)^\alpha |k| + (1+R)^\alpha e^{-R} e^{c|k|} \lesssim_\alpha |k|^{\alpha+1}$ (by taking $R = c|k|$)

Step 5: using F. Series dec.: $\|k_{F(\mathcal{L})}\|_{2,\alpha} \leq \sum_{k \in \mathbb{Z}\setminus\{0\}} |\hat{G}(k)| \, \|k_{e^{ikM}-1}\|_{2,\alpha} \lesssim_\alpha \sum_{k\neq 0} |\hat{G}(k)| \, |k|^{\alpha+1}$

$$\underset{CS}{\lesssim} \left(\sum_{k \in \mathbb{Z}\setminus\{0\}} |\hat{G}(k)|^2 |k|^{2\beta} \right)^{1/2} \left(\sum_k |k|^{2(\alpha+1-\beta)} \right)^{1/2} \underset{\boxed{\beta > \alpha + 3/2}}{\lesssim_{\alpha,\beta}} \|G\|_{L^2_\beta} \simeq \|F\|_{L^2_\beta}$$

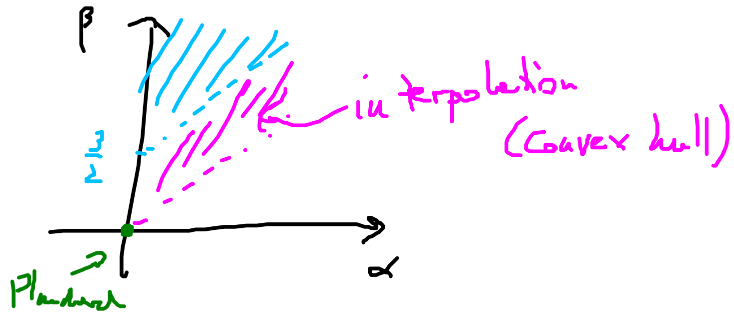
So (*) $\| k_{F(\mathcal{L})} (1+|\cdot|_Z)^{\alpha} \|_2 \lesssim_{\alpha,\beta} \|F\|_{L^2_\beta}$, $\quad \beta > \alpha + \frac{3}{2} \quad (\operatorname{supp} F \subseteq [1/2, 2])$

how to get rid of this?

Step 6: Mauceri-Meda's interpolation trick:

For $\alpha = 0$, $\quad \|K_{F(\mathcal{L})}\|_2^2 \overset{\text{Plancherel}}{=} c \int_0^\infty |F(\lambda)|^2 \lambda^Q \frac{d\lambda}{\lambda} \overset{\operatorname{supp} F \subseteq [1/2,2]}{\simeq} \|F\|^2_{L^2(\mathbb{R})} \quad (+)$

So the estimate (*) is valid for (α, β) in



By interpolation, (*) is also valid for any $\boxed{\beta > \alpha}$.

So we know: $\| k_{F(\mathcal{L})} \|_1 \lesssim_s \| F \|_{L^2_s}$, $s > \frac{Q}{2}$, $(\operatorname{supp} F \subseteq [1/2, 2])$

Question: is $\frac{Q}{2}$ the OPTIMAL threshold?

- In step 1 $(s=1,\ G \simeq \mathbb{R}^n,\ \mathcal{L} = -\Delta)$: Yes.
- In higher step?
  - in general it is still an open question
  - optimal threshold $s_0 = s_0(\mathcal{L})$ known only in a few cases

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Problem of optimal threshold related to — BOCHNER-RIESZ summability (multipliers $(1-t\mathcal{L})_+^{\alpha}$, $t>0$: L^1-bdd for $\alpha > s_0$)

Miyachi-Peral est's for wave eq.:

$\| (1+\mathcal{L})^{-\alpha/2} \cos(t\sqrt{\mathcal{L}}) \|_{1\to 1} \lesssim_{\alpha} (1+|t|)^{\alpha}$: in $\mathbb{R}^n$ this is true for $\boxed{\alpha > \frac{n-1}{2}}$

By subordination: if $F : \mathbb{R} \to \mathbb{C}$ even, $(\operatorname{supp} F \subseteq [-1,1] \setminus (-1/2, 1/2))$, $F(\sqrt{\mathcal{L}}) = G(\sqrt{\mathcal{L}})(1+\mathcal{L})^{-\alpha/2}$, $(G(\lambda) = F(\lambda)(1+\lambda^2)^{\alpha/2})$ then

$$F(\mathcal{L}) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{G}(\tau)(1+\mathcal{L})^{-\alpha/2} \cos(\tau\sqrt{\mathcal{L}})\, d\tau \Rightarrow \| F(\mathcal{L}) \|_{1\to 1} \lesssim \int_{\mathbb{R}} |\hat{G}(\tau)| (1+|\tau|)^{\alpha} d\tau \underset{\text{CS}}{\lesssim_{\alpha,\beta}} \| G \|_{L^2_\beta}, \quad \beta > \alpha + 1/2$$

So in $\mathbb{R}^n$ the Miyachi-Peral est w/ $\frac{n-1}{2}$ is sharp as it implies the sharp L^1 est for multipliers. (w/ $s > \frac{n}{2}$)

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known "trivial" bds: $\frac{r}{2} \le s_0(\mathcal{L}) \le \frac{Q}{2}$ $\quad$ $r = \dim \mathfrak{g}_1$ horizontal rank

constr/H-M. (upper bound); comparison w/ $\mathbb{R}^n$ via transplantation [Kenig-Stanton-Tomas] (lower bound)