Evolution PDEs on Nilpotent Lie Groups and Functional Calculus for Sub-Laplacians

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LECTURE 4

Main References

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The Heisenberg group

$H_m = \underbrace{\mathbb{R}^m_x \times \mathbb{R}^m_y}_{1^{st}\text{ layer}} \times \underbrace{\mathbb{R}_u}_{2^{nd}\text{ layer}}$ $\quad (x,y,u)\cdot(x',y',u') = (x+x', y+y', u+u'+\frac{1}{2}(xy'-x'y))$

$Q = 2m+2$ hom. dim.
$d = 2m+1$ top. dim.
$r = 2m$ horiz. rank

- basis of 1st layer: $X_j = \partial_{x_j} - \frac{y_j}{2}\partial_u$, $Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_u$, $[X_j, Y_j] = \partial_u$
- sub-Laplacian $\mathcal{L} = -\sum_j (X_j^2 + Y_j^2) = \underbrace{-\Delta_{(x,y)} - \frac{|(x,y)|^2}{4}\partial_u^2}_{G\ (\text{"Grushin"})} + \underbrace{\sum_j (x_j \partial_{y_j} - y_j \partial_{x_j})\partial_u}_{A\ (\text{"angular"})}$

(A) invariant under $U(m)\times\{1\}$ acting on $\mathbb{C}^m_{(x,y)} \times \mathbb{R}_u$ $\rightarrow$ kernels $K_{F(\mathcal{L})}(z,u)$ are radial in $z=(x,y)$

(B) commutes w/ $U = -i\partial_u$

(C) partial F.T. in u gives $\mathcal{L}^\mu = -\Delta_z + \frac{\mu^2}{4}|z|^2 + i\mu(x_j\partial_{y_j} - y_j\partial_{x_j})$ Twisted Laplacian (magnetic Schrödinger op.) ($\mu \in \mathbb{R}$ dual to u)

also: $(f*g)^\mu = f^\mu \times_\mu g^\mu$, where $F \times_\mu G(x,y) = \int_{\mathbb{R}^{2m}} F(x',y')\, G(x-x', y-y')\, e^{i\frac{\mu}{2}(xy'-x'y)}\, dx'dy'$ twisted convolution

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For $\mu \neq 0$: $\mathcal{L}^\mu$ has discrete spectrum $\{|\mu|(2n+m)\}_{n\in\mathbb{N}}$ and corr. radial eigenfunctions $e_n^{(m-1)}(|\mu||z|^2)$

where $e_m^{(k)}(t) = e^{-t/4} L_m^{(k)}(t/2)$, $L_m^{(k)}$ Laguerre polynomial

$\rightsquigarrow$ kernel formula: $K^\mu_{F(\mathcal{L})} \doteq \sum_{n\in\mathbb{N}} F(|\mu|(2n+m))\, \underline{|\mu|^m}\, e_n^{(m-1)}(|\mu||z|^2)$

$\rightsquigarrow$ joint F.C. w/ U: $K^\mu_{F(\mathcal{L},U)} \doteq \sum_{n\in\mathbb{N}} F(|\mu|(2n+m), \mu)\, |\mu|^m\, e_n^{(m-1)}(|\mu||z|^2)$

(the normalisation is appropriate as L^1-scaling preserves L^2-op. norm of (twisted) convolution (spectral projection))

From this we can recover the Plancherel formula for $\mathcal{L}$: $\left[\int_0^\infty e_n^{(k)}(t)\, e_{n'}^{(k)}(t)\, t^k\, dt \doteq_k \frac{(n+k)!}{n!} \simeq_k \langle n\rangle^k\right]$

$$\int_G |K_{F(\mathcal{L})}(z,u)|^2\, dz\, du \doteq \int_{\mathbb{R}} \sum_{n\in\mathbb{N}} |F(|\mu|(2n+m))|^2 |\mu|^m \frac{(n+m-1)!}{n!}\, d\mu \doteq \int_0^\infty |F(\lambda)|^2 \underbrace{\left(\sum_{n\in\mathbb{N}} \frac{(n+m-1)!}{n!\,(2n+m)^{m+1}}\right)}_{\simeq \langle n\rangle^{-2} \rightarrow \text{converges!}} \lambda^{m+1} \frac{d\lambda}{\lambda}$$

Hebisch's Trick: $\mathcal{L}^\mu = \underbrace{-\Delta_z + \frac{\mu^2}{4}|z|^2}_{G^\mu} + \underbrace{i\mu(x_j\partial_{y_j} - y_j\partial_{x_j})}_{A^\mu}$, $A^\mu F = 0$ for any radial fctn F on $\mathbb{R}^{2m}$

$\Rightarrow \mathcal{L}^\mu F = G^\mu F$ for F radial

So: if $f(z,u)$ is z-radial, then

$$\|(\mathcal{L}^\mu)^{1/2} f^\mu\|_2^2 = \langle \mathcal{L}^\mu f^\mu, f^\mu\rangle \underset{f^\mu \text{ radial}}{=} \langle G^\mu f^\mu, f^\mu\rangle \underset{\substack{\text{pot energy}\\ \le \text{ energy}}}{\geq} \frac{\mu^2}{4}\langle |z|^2 f^\mu, f^\mu\rangle = \frac{\mu^2}{4}\| |z| f^\mu\|_2^2$$

Rearranging: $\| |z| f^\mu\|_{L^2(\mathbb{R}^{2m})} \le 2\| |\mu|^{-1}(\mathcal{L}^\mu)^{1/2} f^\mu\|_{L^2(\mathbb{R}^{2m})} \underset{\text{Fubini \& Plancherel}}{\rightsquigarrow} \| |z| f\|_{L^2(H_m)} \le 2\| |U|^{-1}(\mathcal{L})^{1/2} f\|_{L^2(H_m)}$ $(U = -i\partial_u)$

$\underset{\text{interpolation}}{\rightsquigarrow} \| |z|^\alpha f\|_{L^2(H_m)} \lesssim_\alpha \| |U|^{-\alpha}(\mathcal{L})^{\alpha/2} f\|_{L^2(H_m)}$, $\alpha\in[0,1]$, f radial

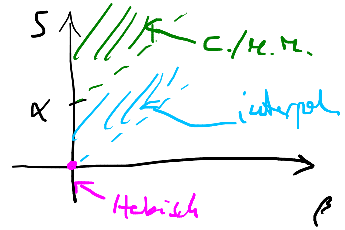
As $k_{F(\mathcal{L})}(z,u)$ is z-radial, we deduce, if $\operatorname{supp} F \subseteq [1/2, 2]$, $F_\alpha(\lambda) = \lambda^{\alpha/2}F(\lambda)$,

$$\| |z|^\alpha k_{F(\mathcal{L})}\|_2^2 \lesssim_\alpha \|k_{|U|^{-\alpha}F_\alpha(\mathcal{L})}\|_2^2 \underset{(Pl.)}{\doteq} \int_{\mathbb{R}}\sum_{n\in\mathbb{N}} |\mu|^{-2\alpha}|F_\alpha(|\mu|(2n+m))|^2 |\mu|^m \frac{(n+m+1)!}{n!}\, d\mu$$

$$\doteq \int_0^\infty |F_\alpha(\lambda)|^2 \sum_{n\in\mathbb{N}} \frac{(n+m-1)!}{n!\,(2n+m)^{m+1-2\alpha}} \lambda^{m+1-2\alpha}\frac{d\lambda}{\lambda} \simeq_\alpha \int_0^\infty |F(\lambda)|^2 \underbrace{\sum_{n\in\mathbb{N}} \langle n\rangle^{-2+2\alpha}}_{\text{converges for } \alpha<\frac{1}{2}\,!} \lambda^{m+1-\alpha}\frac{d\lambda}{\lambda}$$

Thus $\|(1+|z|)^\alpha k_{F(\mathcal{L})}\|_{L^2(G)} \lesssim \|F\|_{L^2(\mathbb{R})}$ for any $\alpha\in[0,1/2)$, $\operatorname{supp} F \subseteq [1/2,2]$

Also: $(*)$ $\|(1+|z|)^\alpha(1+(|z|+|u|)^{1/2})^\beta F(\mathcal{L})\| \lesssim_{\alpha,\beta,s} \|F\|_{L^2_s}$, $s>\alpha+\beta$ (by C./M.M.)



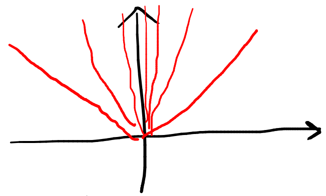
So: $(*)$ by interpol. also holds for $s>\beta$, $\alpha<1/2$ $\rightsquigarrow$ by C.S. this implies IMPROVED L^1-estimate: $\|k_{F(\mathcal{L})}\|_{L^1} \lesssim_s \|F\|_{L^2_s(\mathbb{R})}$, $s > \frac{2m+1}{2}$

This proves: $s_0(\mathcal{L}) \le \underset{\text{(top. dim.)}}{\frac{2m+1}{2}}$ on H_m. (Hebisch '93 / Müller & Stein '94)

Another consequence of kernel formula: no dispersion for Schrödinger eq.

$\mathcal{L}, U$ have joint spectral resolution, w/ spectrum the closure of $\{(|\mu|(2n+m),\mu)\}_{n\in\mathbb{N}}$

In particular, $U^{-1}\mathcal{L}$ has discrete spectrum $\sigma=\{\pm(2n+m)\}_{n\in\mathbb{N},\,\pm}$



(Heisenberg Fan)

and we can decompose $L^2(H_n)=\bigoplus_{\lambda\in\sigma}E_\lambda$ (eigenspace decomposition).

So, for $f\in E_\lambda$, $\lambda\in\sigma$, we have $\mathcal{L}f=\lambda Uf \Rightarrow e^{it\mathcal{L}}f=e^{i\lambda tU}f=e^{\lambda t\partial_u}f=f(z,u+t\lambda)$

$\Rightarrow$ if we pick $f\in\mathcal{S}(G)\cap E_\lambda$ (e.g. $f^\mu=\chi_1(\pm\mu)\,|\mu|^m\,e_n^{(m-1)}(|\mu||\cdot|^2)$) then $\|e^{it\mathcal{L}}f\|_\infty$ is CONSTANT in t
$\to$ NO DISPERSION!

[Bahouri-Gérard-Xu '00]

Compare: in $\mathbb{R}^n$:

heat kernel $K_{e^{-t\Delta}}(x)=\frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/4t}$, Schrödinger kernel $K_{e^{it\Delta}}(x)=\frac{1}{(-4\pi i t)^{n/2}}e^{i|x|^2/4t}$

$$\|K_{e^{it\Delta}}\|_\infty \doteq |t|^{-n/2}\doteq \|e^{it\Delta}\|_{1\to\infty}$$

Generating fctn for Laguerre polynomials: $\sum_{n\in\mathbb{N}} L_n^{(k)}(t)\,w^n=(1-w)^{-k-1}\exp\left(\frac{tw}{w-1}\right)$, $|w|<1$

Then:
$$K^\mu_{e^{-t\mathcal{L}}}(z)\doteq\sum_{n\in\mathbb{N}}e^{-t|\mu|(2n+m)}|\mu|^m e_n^{(m-1)}(|\mu||z|^2)=\sum_{n\in\mathbb{N}}e^{-t|\mu|(2n+m)}|\mu|^m e^{-|\mu||z|^2/4}L_n^{(m-1)}(|\mu||z|^2/2)$$
$$=|\mu|^m e^{-t|\mu|m-|\mu||z|^2/4}\sum_{n\in\mathbb{N}}(e^{-2t|\mu|})^n L_n^{(m-1)}(|\mu||z|^2/2)$$
$$=|\mu|^m e^{-|\mu|(tm+|z|^2/4)}(1-e^{-2t|\mu|})^{-m}\exp\left(\frac{|\mu||z|^2}{2}\,\frac{e^{-2t|\mu|}}{e^{-2t|\mu|}-1}\right)$$
$$\doteq t^{-m}\left(\frac{t|\mu|}{\sinh(t|\mu|)}\right)^m\exp\left(-\frac{1}{4t}\left(\frac{t|\mu|}{\tanh(t|\mu|)}\right)|z|^2\right)$$

Hulanicki-Gaveau Formula for the heat kernel on Heisenberg groups

and $K_{e^{-t\mathcal{L}}}(z,u)\doteq t^{-m}\int_{\mathbb{R}}\left(\frac{t\mu}{\sinh(t\mu)}\right)^m\exp\left(-\frac{1}{4t}\left(\frac{t\mu}{\tanh(t\mu)}\right)|z|^2-iu\mu\right)d\mu$

$\longrightarrow$ note: it immediately implies "Gaussian" decay in z ...

Schrödinger: $K^\mu_{e^{it\mathcal{L}}}(z,\mu)\doteq t^{-m}\left(\frac{t\mu}{\sin(t\mu)}\right)^m\exp\left(-\frac{i}{4t}\left(\frac{t\mu}{\tan(t\mu)}\right)|z|^2\right)$ singular at $t\mu\in\pi\mathbb{Z}\setminus\{0\}$ ($\Rightarrow K_{e^{it\mathcal{L}}}$ has C-Z singularities at $(0,\pm t(2n+m))$, [Sikora-Zienkiewicz '02])

General 2-step groups:

$G \cong \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\quad (x,u)\cdot(x',u') = (x+x', u+u'+\frac{1}{2}[x,x'])$

l.i. v.f's: $X_j = \partial_{x_j} + \frac{1}{2}[x,e_j]\cdot\nabla_u$, $\mathcal{L} = -\sum_j X_j^2$ $\quad \mu[x,x'] = \langle J_\mu x, x'\rangle$ $\quad J_\mu : \mathfrak{g}_1 \to \mathfrak{g}_1$ skewsymm. $\forall \mu \in \mathfrak{g}_2^*$

$$X_j^\mu = \partial_{x_j} + \frac{i}{2}(J_\mu x)_j \rightsquigarrow \mathcal{L}^\mu = \underbrace{-\Delta_x + \frac{1}{4}|J_\mu x|^2}\ \underbrace{-i(J_\mu x)\cdot\nabla_x} =: \mathcal{G}^\mu + \Lambda^\mu$$

If we write $\sqrt{-J_\mu^2} = \sum_j b_j^\mu P_j^\mu$ (sp. dec.) then $\mathcal{G}^\mu = \sum_j \left(-\Delta_{P_j x} + \frac{1}{4}|b_j^\mu|^2 |P_j^\mu x|^2\right)$

Assuming $\ker J_\mu = \{0\}$ we have eigenvalues $\sum_j |b_j^\mu|(2n_j + r_j)$, $2r_j = \mathrm{rk}\, P_j^\mu$, $n = (n_j)_j \in \mathbb{N}^M$

eigenfcts $\prod_j |b_j^\mu|^{r_j} e_{n_j}^{(r_j-1)}(b_j^\mu |P_j^\mu x|^2)$

$\rightsquigarrow$ kernel formula $K^\mu_{F(\mathcal{L})}(x) \doteq \sum_{n\in\mathbb{N}^M} F\left(\sum_j |b_j^\mu|(2n_j+r_j)\right) \prod_j |b_j^\mu|^{r_j} e_{n_j}^{(r_j-1)}(b_j^\mu |P_j^\mu x|^2)$

[in general: $K^\mu_{F(\mathcal{L})}(x) \doteq \int_{R(P_0)} F\left(\sum_j |b_j^\mu|(2n_j+r_j) + |\eta|^2\right) \prod_j |b_j^\mu|^{r_j} e_{n_j}^{(r_j-1)}(b_j^\mu|P_j^\mu x|^2)\, e^{i\langle \eta, P_0^\mu x\rangle}\, d\eta$]

Heat kernel formula: $K^\mu_{e^{-t\mathcal{L}}}(x) \doteq t^{-\dim \mathfrak{g}_1/2} \det{}^{1/2}(S(tJ_\mu)) \exp\left(-\frac{1}{4t}\langle T(tJ_\mu)x,x\rangle\right)$, $\quad S(z) = \frac{z}{\sin z}$, $T(z) = \frac{z}{\tan z}$

(Hulanicki/Gaveau/Cygan/...) $\quad$ (note: J_μ skewsymm. $\Rightarrow$ iJ_μ selfadj, $(iJ_\mu)^2 = -J_\mu^2$)

H-type groups: $|J_\mu x| = |\mu|\,|x| \Rightarrow M = 1$, $b_1^\mu = |\mu|$ $\rightarrow$ Formulas are fully analogous to those on Heisenberg groups

In this context:

- Hebisch's trick extends, leading to $s_0(\mathcal{L}) \le \frac{\dim G}{2}$ $\forall$ H-type group.
- $|U|^{-1}\mathcal{L}$ has discrete spectrum $(2n + \dim\mathfrak{g}_1/2)_n$ $\rightsquigarrow$ $e^{it\mathcal{L}} = e^{it\lambda|U|}$ $\longleftarrow$ partial wave prop. on $\mathfrak{g}_2 \cong \mathbb{R}^{\dim \mathfrak{g}_2}$
  $(U = -i\nabla_u,\ |U| = \sqrt{-\Delta_u})$ $\quad$ on λ-eigenspace $\rightarrow$ can expect dispersion (Del Hierro '05) like for wave eq.
- Müller & Stein '99 / Müller & Seeger '15: Miyachi–Peral est's $\|(1+\mathcal{L})^{-\alpha/2}\cos(\sqrt{\mathcal{L}})\|_{p\to p} < \infty$ for $\alpha > \frac{d-1}{2}\left|\frac{1}{2}-\frac{1}{p}\right|$

Arbitrary 2-step groups: $\frac{d}{2} \le s_0(\mathcal{L}) < \frac{Q}{2}$ (M. & Müller, GAFA '16) (also $s_0(\mathcal{L}) = d/2$ on many examples beyond H-type ...)

Q: What about higher step?