Singular Integral Operators with Flag Kernels

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In the five lectures we will introduce the class of singular integral operators $T: f \mapsto f * \mathcal{K}$ with flag kernel $\mathcal{K}$ on a homogeneous nilpotent Lie group G with Lie algebra $\mathfrak{g}$ and with dimension N.

We will be mainly concerned with the introduction of techniques and discussion of approaches from A. Nagel, F. Ricci, E. M. Stein and S. Wainger [1, 2, 3]. The goal is to show that this class of operators T form an algebra under composition and that these operators T are bounded on $L^p(G)$ for $1<p<\infty$.

References

[1] A. Nagel, F. Ricci, E. Stein: Harmonic analysis and fundamental solutions on nilpotent Lie groups. In Analysis and partial differential equations, 249-275. Lecture Notes in Pure and Appl. Math., 122. Dekker, New York, 1990.

[2] A. Nagel, F. Ricci, E. Stein, S. Wainger: Singular integrals with flag kernels on homogeneous groups, I. Rev. Mat. Iberoam. 28 (2012), 631-722.

[3] ——, Algebras of singular integral operators with kernels controlled by multiple norms. Mem. Amer. Math. Soc. 256 (2018), vii+141 pp.

Outlines

1. Motivation
2. Flag kernels on $\mathbb{R}^N$
3. L^p boundedness of the op. $T_{\mathcal{K}}: f \mapsto f * \mathcal{K}$ with flag kernel $\mathcal{K}$ on G, $1<p<\infty$
4. Composition of two operators $T_{\mathcal{K}_j}: f \mapsto f * \mathcal{K}_j$, $j=1,2$.

1. Motivation

In this section let $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ denote a smooth Calderón-Zygmund kernel in Euclidean space $\mathbb{R}^N$. That is, $\mathcal{K}$ is C^∞ away from the origin and s.t. the following size estimates and cancellation conditions:

(i) For every $\alpha \in \mathbb{N}_0^N$ there exists a constant $C_\alpha > 0$ such that when away away from the origin

$$|\partial^\alpha \mathcal{K}(x)| \leq C_\alpha |x|^{-N-|\alpha|} \tag{1.1}$$

(ii) For any $\psi \in C_c^\infty(\mathbb{R}^N)$ and any $R > 0$, there exists a constant A independent of R such that

$$|\langle \mathcal{K}, \psi_R \rangle| \leq A < \infty. \tag{1.2}$$

Here, $\psi_R(x) = \psi(R \cdot x) = \psi(Rx_1, \cdots, Rx_N)$.

<u>Question 1</u> How do $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ satisfying (1.1) come about?

In P.D.E., differential op. $a(x,D) = \sum_{|\alpha|\leq m} b_\alpha(x) D_x^\alpha$ has a symbol $a(x,\xi) = \sum_{|\alpha|\leq m} b_\alpha(x)\xi^\alpha$. It is known that symbolic calculus is an effective way to study P.D.E. For instance, if $a = a(x,\xi) \in S^m$, i.e. a is C^∞ w.r.t. x and ξ and s.t. $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim \langle \xi \rangle^{m-|\alpha|}$ $\forall \alpha, \beta \in \mathbb{N}_0^n$, then, $a(x,D)$ is bounded from $H^s(\mathbb{R}^N)$ to $H^{s-m}(\mathbb{R}^N)$.

Drawback for symbolic calculus: require ξ to be of high regularity

Smoothness requirement for ξ $\rightsquigarrow$ retain the local behavior of the op, but eliminate a variety of problems for large x.

Microlocal analysis: localize x and ξ.

Roughly speaking, under appropriate conditions,

$$a = a(\xi) \xrightarrow{1-1} \check{a} = \mathcal{K} \in \mathcal{S}'(\mathbb{R}^N) \longrightarrow \text{op. } T: f \mapsto f * \mathcal{K}$$

(multiplier) (singular integral op.)

<u>Example</u> Suppose that $m = m(\xi)$ is a C^∞ function on $\mathbb{R}^N \setminus \{0\}$ that is homogeneous of degree zero. Then, there exist $b \in \mathbb{C}$ and a function $\Omega \in C^\infty(S^{N-1})$ with $\int_{S^{N-1}} \Omega = 0$ such that

$$\check{m} = b\,\delta_0 + \text{p.v.} \frac{\Omega(x/|x|)}{|x|^N}.$$

Note that such m s.t.

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}.$$

<u>Fact 1</u> Suppose $m \in L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$. Then,

(i) If m s.t.

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}_0^N$$

then $\check{m}$ agrees with a smooth function K when away from the origin and s.t.

$$|\partial_x^\alpha K(x)| \le C_\alpha |x|^{-N-|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}_0^n$$

(ii) If m s.t.

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}_0^n \text{ with } 0 \le |\alpha| \le [\tfrac{N}{2}] + 1$$

then $\check{m}$ agrees with a locally integrable function K when away from the origin and s.t.

$$\sup_{y \in \mathbb{R}^N \setminus \{0\}} \int_{|x| \ge 2|y|} |K(x-y) - K(x)|\, dx \le A < \infty$$

Question 2 How do $\mathcal{K}$ satisfying (1.2) come about?

Fact 2 Suppose $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ equals with a function K when away from the origin that

$$|\partial^\alpha K(x)| \le C_\alpha |x|^{-N-|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| = 1.$$

Then, $\hat{\mathcal{K}}$ is bounded iff for all $\psi \in C_c^\infty(\mathbb{R}^N)$ normalized bump functions and for all $R > 0$ such that

$$|\langle \mathcal{K}, \psi_R \rangle| \lesssim 1$$

where $\psi_R(x) = \psi(R \cdot x)$, $R \cdot x = (Rx_1, \cdots, Rx_N)$

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2. Flag kernels on $\mathbb{R}^N$

Müller-Ricci-Stein ('95): Marcinkiewicz multiplier on Heisenberg group $\mathbb{H}_n$.

Let $\mathfrak{h}_n$ denote the Lie algebra of $\mathbb{H}_n$. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n, T$ be the basis of $\mathfrak{h}_n$ ∋

$$\mathbb{H}_n \ni (x+iy, t) = \exp\left(x_1 X_1 + \cdots + x_n X_n + y_1 Y_1 + \cdots + y_n Y_n + tT\right)$$

Then, $[X_i, Y_i] = 4T$. Set $\mathcal{L} := -\sum_{j=1}^n (X_j^2 + Y_j^2)$, the sub-Laplacian on $\mathbb{H}_n$.

Let $dE_{\mathcal{L}}$ and dE_{iT} be the spectral measures of $\mathcal{L}$ and iT respectively, i.e.

$$\mathcal{L} = \int_0^\infty \xi \, dE_{\mathcal{L}}(\xi), \qquad iT = \int_{\mathbb{R}} \eta \, dE_{iT}(\eta)$$

Note that $\mathcal{L}$ and iT commute, so do their spectal measures $dE_{\mathcal{L}}(\xi)$ and $dE_{iT}(\eta)$.

Suppose $m(\xi, \eta)$ is a bounded function on $\mathbb{R}_+ \times \mathbb{R}$. Then the joint multiplier $m(\mathcal{L}, iT)$ is given by

$$m(\mathcal{L}, iT) = \int_{\{(\xi,\eta):\, \xi > 0,\ \eta \in \mathbb{R}\}} m(\xi, \eta) \, dE_{\mathcal{L}}(\xi) \, dE_{iT}(\eta).$$

Consider now multiplier $m = m(\xi, \eta)$ on $\mathbb{R}_+ \times \mathbb{R}$ s.t.

$$\left| (\xi \partial_\xi)^\alpha (\eta \partial_\eta)^\beta m(\xi, \eta) \right| \leq C_{\alpha\beta} \tag{2.1}$$

This multiplier is invariant under two parameter groups of dilation

$$(\xi, \eta) \longmapsto (\lambda_1 \xi, \lambda_2 \eta) \quad \text{for } \lambda_1, \lambda_2 > 0.$$

But on $\mathbb{H}_n$ there is no two parameter group of auto morphic group. Instead, the automorphic dilation on $\mathbb{H}_n$ is

$$\mathbb{H}^n \ni (z, t) \longmapsto (\lambda z, \lambda^2 t)$$

Very amazingly, Müller, Ricci, and Stein got the boundedness of $m(\mathcal{L}, iT)$ on $L^p(G)$ for $1<p<\infty$ and showed that its convolution kernel has some special singularity construction.

<u>Thm</u> (Müller-Ricci-Stein, 1995) Suppose that m s.t. Marcinkiewicz condition (2.1) for all $\alpha, \beta \leq N$, $N >> 1$. Then $m(\mathcal{L}, iT)$ is bounded on $L^p(G)$ for $1<p<\infty$.

<u>Thm</u> (Müller-Ricci-Stein, 1995) Suppose that m s.t. Marcinkiewicz condition (2.1) for all α and β. Then the convolution kernel $\mathcal{K}$ of $m(\mathcal{L}, iT)$ is smooth away from $z=0$, radial in $t=0$, and s.t.

(i) (size estimates) $\left| \partial_z^\gamma \partial_t^j \mathcal{K}(z,t) \right| \lesssim_{\gamma, j} |z|^{-2n-|\gamma|} (|z|^2 + t)^{-1-j}$

for all $\gamma \in \mathbb{N}_0^{2n}$ and $j \in \mathbb{N}_0$ when away from $z = 0$

(ii) (cancellation conditions)

$$\left| \int_{\mathbb{R}} \partial_z^\gamma \mathcal{K}(z,t)\, \varphi(\lambda t)\, dt \right| \lesssim_\gamma |z|^{-2n-|\gamma|}$$ for every $\gamma \in \mathbb{N}_0^{2n}$, every n.b.f. φ on $\mathbb{R}$, and every $\lambda > 0$,

$$\left| \int_{\mathbb{R}^2} \partial_t^j \mathcal{K}(z,t)\, \psi(\lambda z)\, dz \right| \lesssim_j |t|^{-1-j}$$ for every $j \in \mathbb{N}_0$, every n.b.f. ψ on $\mathbb{C}^n$, and every $\lambda > 0$,

$$\left| \int_{\mathbb{H}_n} \partial_z^\gamma \partial_t^j \mathcal{K}(z,t)\, \eta(\lambda_1 z, \lambda_2 t)\, dz\, dt \right| \lesssim_{\gamma, j} 1$$ for every $(\gamma, j) \in \mathbb{N}_0^{2n} \times \mathbb{N}_0$, every n.b.f. η on $\mathbb{H}_n$, and every $\lambda_1, \lambda_2 > 0$

Remark In the region where $|t| < |z|^2$, the kernel $\mathcal{K}$ s.t. Calderón-Zygmund type

$|t| > |z|^2$, $\mathcal{K}$ would be that of singular product kernel

We do see the coexistence of a one-parameter structure on one part of the space and a two-parameter structure on the other part of the space.

Remark The kernel $\mathcal{K}$ has singularity supported on an increasing subspace

$$(0) \subset \mathbb{R} \subset \mathbb{C}^n \times \mathbb{R} = \mathbb{H}_n$$

Flag kernels on $\mathbb{R}^N$

Let G be a homogeneous nilpotent Lie group with Lie algebra $\mathfrak{g}$ and with dim N. Since $\mathbb{R}^N \simeq \mathfrak{g} \simeq G$, with an appropriate choice of coordinates we may assume that $G = \mathbb{R}^N$ and the automorphic dilates are given by

$$\lambda \cdot \mathbf{x} = (\lambda^{d_1} x_1, \cdots, \lambda^{d_N} x_N) \quad \text{with } 0 < d_1 \le d_2 \le \dots \le d_N$$

Consider a partition

$$N = a_1 + a_2 + \dots + a_n \qquad \text{with each } a_\ell \in \mathbb{N}$$

and write $\mathbb{R}^N$ as

$$\mathbb{R}^N = \mathbb{R}^{a_1} \oplus \dots \oplus \mathbb{R}^{a_n}$$

Then $\mathbf{x} = (x_1, \cdots, x_N) \in \mathbb{R}^N$ can be written as $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ with $\mathbf{x}_\ell = (x_{p_\ell}, \cdots, x_{q_\ell}) \in \mathbb{R}^{a_\ell}$, $q_\ell - p_\ell + 1 = a_\ell$. The standard flag $\mathcal{F}$ in $\mathbb{R}^N$ associated to this partition is

$$\mathcal{F}: \quad (0) \subset \mathbb{R}^{a_n} \subset \mathbb{R}^{a_{n-1}} \oplus \mathbb{R}^{a_n} \subset \dots \subset \mathbb{R}^{a_2} \oplus \dots \oplus \mathbb{R}^{a_n} \subset \mathbb{R}^{a_1} \oplus \dots \mathbb{R}^{a_n} = \mathbb{R}^N$$

Given positive numbers $0 < \lambda_1 \le \dots \le d_N$, recall that we have the family of dilations on $\mathbb{R}^N$

$$\lambda \cdot \mathbf{x} = (\lambda^{d_1} x_1, \cdots, \lambda^{d_N} x_N).$$

Let $|\mathbf{x}|$ be a smooth homogeneous norm on $\mathbb{R}^N$ so that $|\lambda \cdot \mathbf{x}| = \lambda |\mathbf{x}|$. Then,

- the dilation action on $\mathbb{R}^{a_\ell}$ is $\lambda \cdot \mathbf{x}_\ell = (\lambda^{d_{p_\ell}} x_{p_\ell}, \cdots, \lambda^{q_\ell} x_{q_\ell})$
- the homogeneous dimension of $\mathbb{R}^{a_\ell}$ is $Q_\ell = d_{p_\ell} + \dots + d_{q_\ell}$
- $\exists$ a homogeneous norm N_ℓ on $\mathbb{R}^{a_\ell}$, $N_\ell(\mathbf{x}_\ell) \approx \sup_{p_\ell \le j \le q_\ell} |x_j|^{1/d_j}$, $N_\ell(\lambda \cdot \mathbf{x}_\ell) = \lambda N_\ell(\mathbf{x}_\ell)$ [6]

Let $\{1, \cdots, n\} = A \cup B$ with $A = \{\ell_1, \cdots, \ell_r\}$, $B = \{m_1, \cdots, m_s\}$, $A \cap B = \phi$.

Denote

$$N_a = a_{\ell_1} + \cdots + a_{\ell_r}, \qquad N_b = a_{m_1} + \cdots + a_{m_s}.$$

Then we can write $x \in \mathbb{R}^N$ as

$$x = (x_A, x_B) \in \mathbb{R}^{N_a} \times \mathbb{R}^{N_b} \quad \text{with } x_A = (x_{\ell_1}, \cdots, x_{\ell_r}), \ x_B = (x_{m_1}, \cdots, x_{m_s})$$

Def. A distribution $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ is a flag kernel adapted to the flag

$$\mathcal{F}: \quad (0) \subset \mathbb{R}^{a_n} \subset \mathbb{R}^{a_{n-1}} \oplus \mathbb{R}^{a_n} \subset \cdots \subset \mathbb{R}^{a_2} \oplus \cdots \oplus \mathbb{R}^{a_n} \subset \mathbb{R}^N$$

if $\mathcal{K}$ is C^∞ away from $x_1 = 0$ and s.t. the following size estimates and cancellation conditions

(i) (Size estimates) For every $\alpha = (\bar{\alpha}_1, \cdots, \bar{\alpha}_n) \in \mathbb{N}_0^N$ there exists a constant $C_\alpha > 0$ such that for $x_1 \neq 0$,

$$|\partial^\alpha \mathcal{K}(x)| \leq C_\alpha \prod_{j=1}^{n} \left(N_j(x_1) + \cdots + N_j(x_j) \right)^{-Q_j - [\![\bar{\alpha}_j]\!]}$$

where $\bar{\alpha}_j = (\alpha_{p_j}, \cdots, \alpha_{q_{t_j}})$ and $[\![\bar{\alpha}_j]\!] = d_{p_j}\alpha_{p_j} + \cdots + d_{q_{t_j}} \alpha_{q_{t_j}}$.

(ii) (Cancellation conditions) For any $\psi \in C_c^\infty(\mathbb{R}^{N_b})$ and any positive numbers $R_1, \cdots, R_s$, the distribution $\mathcal{K}^\#_{\psi, R} \in \mathcal{S}'(\mathbb{R}^{N_a})$ defined by

$$\langle \mathcal{K}^\#_{\psi,R}, \varphi \rangle = \langle \mathcal{K}, \psi_R \otimes \varphi \rangle \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^{N_a})$$

s.t. the size estimates of part (i). Here, $\psi_R(x_B) = \psi(R_1 \cdot x_{m_1}, \cdots, R_s \cdot x_{m_s})$.

Remark The first example of flag kernel is due to Müller-Ricci-Stein ('95).
The convolution kernel of $m(\mathcal{L}, iT)$ is a flag kernel when m s.t. Marcinkiewicz condition (2.1)

- partition: $2n+1 = (2n) + 1$, $\quad \mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$,

homogeneous norm: $\mathbb{H}_n \ni (z, t)$, $\quad N_1(z) \sim |z|$, $\quad N_2(t) \sim |t|^{1/2}$

homogeneous degree: $Q_1 = 2n$, $\quad Q_2 = 2$

thus, $|z|^{-2n-|\gamma|} (|z|^2 + t)^{-1-j} \approx N_1(z)^{-Q_1 - |\gamma|} \left(N_1(z) + N_2(t) \right)^{-Q_2 - 2j}$

- the convolution $\mathcal{K}$ of the joint multiplier $m(\mathcal{L}, iT)$ is relative to the flag

$$\mathcal{F}: \ (0) \subset \mathbb{R} \subset \mathbb{C}^n \times \mathbb{R} = \mathbb{H}_n.$$

More examples of flag kernels

① Let η be a smooth even function with compact support in $\mathbb{R}^2$. Then the distribution given by integration against the function

$$K_1(x,y) = \frac{1}{xy}\,\eta(y/x) \qquad \text{on } \{y \neq 0\}$$

is a flag kernel adapted to the flag $\{(0,0)\} \subset \{(x,0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$.

② A distribution on $\mathbb{R}^3$ given by integration against the function

$$\frac{\operatorname{sgn}(x_2) \quad \operatorname{sgn}(x_3)}{x_1^2 \sqrt{x_1^2 + x_2^2}\ \sqrt{x_1^2 + x_2^2 + x_3^2}} \qquad \text{on } \{x_1 \neq 0\}$$

is a flag kernel on $\mathbb{R}^3$ associated to the flag

$$\{(0,0,0)\} \subset \{(0,0,x_3) \mid x_3 \in \mathbb{R}\} \subset \{(0,x_2,x_3) \mid x_2, x_3 \in \mathbb{R}\} \subset \mathbb{R}^3$$

Fourier transform duality of flag kernels and flag multipliers

Let $(\mathbb{R}^N)^*$ be the space of linear functionals on $\mathbb{R}^N$. For any subspace $V \subset \mathbb{R}^N$, let

$$V^\perp = \{ f \in (\mathbb{R}^N)^* \mid f = 0 \text{ on } V \}.$$

Def If $\mathcal{F}$ is a flag on $\mathbb{R}^N$ given by

$$\mathcal{F}: (0) \subset \mathbb{R}^{a_n} \subset \mathbb{R}^{a_{n-1}} \oplus \mathbb{R}^{a_n} \subset \cdots \subset \mathbb{R}^{a_2} \oplus \cdots \oplus \mathbb{R}^{a_n} \subset \mathbb{R}^N$$

its dual flag, denoted by $\mathcal{F}^*$ is

$$\mathcal{F}^*: (0)^\perp \supset (\mathbb{R}^{a_n})^\perp \supset (\mathbb{R}^{a_{n-1}} \oplus \mathbb{R}^{a_n})^\perp \supset \cdots \supset (\mathbb{R}^{a_2} \oplus \cdots \oplus \mathbb{R}^{a_n})^* \supset (\mathbb{R}^N)^\perp$$

The family of dilations on $(\mathbb{R}^N)^*$ can be induced from the family of dilations defined on $\mathbb{R}^N$ so that

$$\langle \lambda \cdot \mathbf{x}, \xi \rangle = \langle \mathbf{x}, \lambda \cdot \xi \rangle.$$

We can write $\xi \in (\mathbb{R}^N)^*$ as $\xi = (\xi_1, \cdots, \xi_n)$ with $\xi_j \in (\mathbb{R}^{a_j})^*$. Let $|\xi|$ be a smooth homogeneous norm on $(\mathbb{R}^N)^*$ and let $|\xi_j|$ be the restriction of the norm on $(\mathbb{R}^{a_j})^*$.

Def A function $m = m(\xi)$ is said to be a flag multiplier relative to the flag

$$\mathcal{F}^*: (0)^\perp \supset (\mathbb{R}^{a_n})^\perp \supset (\mathbb{R}^{a_{n-1}} \oplus \mathbb{R}^{a_n})^\perp \supset \cdots \supset (\mathbb{R}^{a_2} \oplus \cdots \oplus \mathbb{R}^{a_n})^\perp \supset (\mathbb{R}^N)^\perp$$

if m is C^∞ away from the subspace $\xi_n = 0$, and for any $\alpha = (\bar{\alpha}_1, \cdots, \bar{\alpha}_n) \in \mathbb{N}_0^N$ there exists a constant $C_\alpha > 0$ such that

$$\left| \partial_{\xi_1}^{\bar{\alpha}_1} \cdots \partial_{\xi_n}^{\bar{\alpha}_n} m(\xi) \right| \le C_\alpha \prod_{j=1}^{n} \left(|\xi_j| + \dots + |\xi_n| \right)^{-\| \bar{\alpha}_j \|}.$$

<u>Thm</u> (Nagel-Ricci-Stein, 2001)

A distribution $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ is a flag kernel relative to the flag $\mathcal{F}$ in $\mathbb{R}^N$ if and only if its Fourier transform is a flag multiplier relative to the dual flag $\mathcal{F}^*$.

<u>Pf</u> It follows by the induction on the numbers n of steps in the flag.

If $n = 1$, flag kernel $\mathcal{K}$ is a smooth Calderón-Zygmund kernel. ✓

<u>Def</u> (Coarser partitions & coarser flags)

(i) Let $A = (a_1, \cdots, a_n)$ and $B = (b_1, \cdots, b_m)$ be two partitions of N so that

$$N = a_1 + \dots + a_n = b_1 + \dots + b_m$$

The <u>partition</u> B is called <u>coarser</u> than A (or A is <u>finer</u> than B) if

$$b_k = \sum_{j=\tau_k}^{\tau_{k+1}-1} a_j$$

for some integers $1 = \tau_1 < \tau_2 < \cdots < \tau_{m+1} = n+1$

(ii) Let $\mathcal{F}_A$ and $\mathcal{F}_B$ be two flags corresponding to the two partitions A and B of N, respectively. If B is coarser than A, then we say that the <u>flag</u> $\mathcal{F}_B$ is <u>coarser</u> than $\mathcal{F}_A$ (or $\mathcal{F}_A$ is <u>finer</u> than $\mathcal{F}_B$)

Choose a function $\mathcal{J} \in C_c^\infty(\mathbb{R})$ supported in $[\frac{1}{2}, 4]$ such that

$$\sum_{j\in\mathbb{Z}} \mathcal{J}(2^j t) \equiv 1 \quad \text{for all } t>0$$

Denote $\mathcal{E}_n = \{ I = (i_1, \cdots, i_n) \in \mathbb{Z}^n \mid i_1 \le i_2 \le \dots \le i_n \}$. For each $I \in \mathcal{E}_n$, set

$$\mathcal{J}_I(\xi) = \mathcal{J}(2^{i_1}|\xi_1|) \cdots \mathcal{J}(2^{i_n}|\xi_n|).$$

Then we shall have the following result

Prop. (Nagel-Ricci-Stein-Wainger, 2012) Let m be a flag multiplier relative to the flag

$$\mathcal{F}^*: (0)^\perp \supset (\mathbb{R}^{a_n})^\perp \supset \dots \supset (\mathbb{R}^{a_2} \oplus \dots \oplus \mathbb{R}^{a_n})^\perp \supset (\mathbb{R}^N)^\perp$$

Then we have the following decomposition of m

$$m(\xi) = m_0(\xi) \sum_{I\in\mathcal{E}_n} \mathcal{J}_I(\xi) + \sum_{k=1}^{n} m_k(\xi),$$

where m_0 is the Fourier transform of a flag kernel adapted to the flag $\mathcal{F}$, and for each $1 \le k \le n$, m_k is the Fourier transform of a flag kernel adapted to a flag strictly coarser than $\mathcal{F}$.

Pf Let $\theta \in C_c^\infty(\mathbb{R}_+)$ be supported in $\{0<t\le 20\}$ and $\theta(t) \equiv 1$ when $0<t\le 10$

Write

$$m(\xi) = m(\xi)\left(1-\theta(|\xi_{n-1}|\,|\xi_n|^{-1})\right) + m(\xi)\,\theta(|\xi_{n-1}|\,|\xi_n|^{-1})$$
$$=: n_1(\xi) + m_1(\xi)$$

⇝ $n_1(\xi)$ and $m_1(\xi)$ are flag multipliers relative to the flag $\mathcal{F}^*$

& $\check{m}_1$ is a flag kernel relative to a flag coarser than $\mathcal{F}$.

& on supp $\mathcal{J}_1$, $|\xi_{n-1}| \gtrsim 10|\xi_n|$

Next write

$$n_1(\xi) = n_1(\xi)\left(1-\theta(|\xi_{n-2}|\,|\xi_{n-1}|^{-1})\right) + n_1(\xi)\,\theta(|\xi_{n-2}|\,|\xi_{n-1}|^{-1})$$
$$=: n_2(\xi) + m_2(\xi)$$

$\rightsquigarrow$ $n_2(\xi)$ and $m_2(\xi)$ are flag multipliers relative to the flag $\mathcal{F}^*$,

- $\check{m}_2$ is a flag kernel relative to a flag coarser than $\mathcal{F}$.
- on $\operatorname{supp} \eta_2$, $|\xi_{n-2}| \geq 10 |\xi_{n-1}|$.

Proceed inductively one has that

$$m(\xi) = m_0(\xi) + \sum_{k=1}^{n} m_k(\xi)$$

where for each $1 \leq k \leq n$, m_k is the Fourier transform of a flag kernel relative to a flag coarser than $\mathcal{F}$ and $m_0(\xi) = n_n(\xi)$ is supported in $\{\xi = (\xi_1, \cdots, \xi_n) : |\xi_j| \geq 10 |\xi_{j+1}|, 1 \leq j \leq n-1\}$

Since η is supported in $[\frac{1}{2}, 4]$ $\rightsquigarrow$ $m_0(\xi) = \sum_{I \in \mathcal{E}_n} m_0(\xi) \eta_I(\xi)$. $\square$.

We shall characterize flag kernels in terms of a sum of dilates of normalized bump functions with cancellations.

Def

(1) Let $\varphi, \Phi \in C_c^\infty(\mathbb{R}^N)$. We say that Φ is normalized in terms of φ if there exist constants C, $C_m > 0$ and $s_m \in \mathbb{N}$ so that

(i) if φ is supported in the ball $B_r = \{x \in \mathbb{R}^N : |x| < r\}$, then Φ is supported in the ball B_{Cr}

(ii) For every $m \in \mathbb{N}_0$, $\|\Phi\|_{(m)} := \sup_{\substack{|\alpha| \leq m \\ x \in \mathbb{R}^N}} |\partial_x^\alpha \Phi(x)| \leq C_m \|\varphi\|_{(m+s_m)}$.

(2) Let $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^N)$. We say that Φ is normalized in terms of φ if for every $M \in \mathbb{N}_0$ there are constants $C_M > 0$ and $S_M \in \mathbb{N}_0$ such that

$$\|\Phi\|_M := \sup_{\substack{|\alpha| + \beta \leq M \\ x \in \mathbb{R}^N}} (1 + |x|_e)^\beta |\partial_x^\alpha \Phi(x)| \leq C_M \|\varphi\|_{M+S_M}.$$

Recall that in the context of the decomposition

$$\mathbb{R}^N = \mathbb{R}^{a_1} \oplus \dots \oplus \mathbb{R}^{a_n}$$

we can write $\mathbf{x} \in \mathbb{R}^N$ as $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ with $\mathbf{x}_\ell = (x_{p_\ell}, \cdots, x_{q_\ell}) \in \mathbb{R}^{a_\ell}$.

Strong and weak cancellation

Def. We say that the function $\varphi \in \mathcal{S}(\mathbb{R}^N)$ has strong cancellation if

$$\int_{\mathbb{R}^{a_\ell}} \varphi(\mathbf{x}_1, \cdots, \mathbf{x}_n)\, d\mathbf{x}_\ell = 0 \qquad \text{for any } 1 \le \ell \le n$$

i.e. φ has cancellation in each collection of variables $\{x_{p_\ell}, x_{p_\ell+1}, \cdots, x_{q_\ell}\}$ for $1 \le \ell \le n$.

One valuable observation is that cancellation in certain collections of variables is equivalent to the existence of some primitives.

Prop. (Nagel - Ricci - Stein - Wainger, 2012)

Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and let $J_\ell = \{p_\ell, p_\ell+1, \cdots, q_\ell\} \subset \{1, \cdots, N\}$ be non-empty subsets for $1 \le \ell \le r$ and $\{J_\ell\}_{1\le \ell \le r}$ be mutually disjoint. Then,

(i) for $1 \le \ell \le r$, $\displaystyle\int_{\mathbb{R}^{a_\ell}} \psi(\mathbf{x})\, d\mathbf{x}_\ell = 0$

$\Updownarrow$

(ii) there are functions $\Phi_{j_1, \cdots, j_r} \in \mathcal{S}(\mathbb{R}^N)$ normalized w.r.t. φ such that

$$\varphi(\mathbf{x}) = \sum_{j_1 \in J_1} \cdots \sum_{j_r \in J_r} \partial_{j_1} \cdots \partial_{j_r} \Phi_{j_1, \cdots, j_r}(\mathbf{x}).$$

Pf (ii) $\Rightarrow$ (i): by Fundamental Thm of Calculus

(i) $\Rightarrow$ (ii): by induction on n from the lemma as follows

Lemma(1) Let $\psi \in C_c^\infty(\mathbb{R}^N)$. If $\int_{\mathbb{R}^{a_1}} \psi(\mathbf{x})\, d\mathbf{x}_1 = \int_{\mathbb{R}^{a_2}} \psi(\mathbf{x})\, d\mathbf{x}_2 = 0$, then, for each $j \in J_1$, there exist functions $\{\Psi_j\} \subset C_c^\infty(\mathbb{R}^N)$ normalized in terms of ψ so that

(i) $\psi = \sum_{j \in J_1} \partial_j \Psi_j$,

(ii) $\int_{\mathbb{R}^{a_2}} \Psi_j(x)\, d\mathbf{x}_2 = 0$ for each $j \in J_1$.

(2) If $\psi \in \mathcal{S}(\mathbb{R}^N)$, we have the same conclusions except that these functions ψ_j are in $\mathcal{S}(\mathbb{R}^N)$.

Def. Let $\varepsilon > 0$ and $I \in \mathcal{E}_n$. We say that $\varphi \in C_c^\infty(\mathbb{R}^N)$ (resp. $\varphi \in \mathcal{S}(\mathbb{R}^N)$) has weak cancellation with parameter ε and multi-index I if there exist functions $\varphi_{A,B,j_1,\cdots,j_r} \in C_c^\infty(\mathbb{R}^N)$ (resp. $\varphi_{A,B,j_1,\cdots,j_r} \in \mathcal{S}(\mathbb{R}^N)$) normalized in terms of φ, ∋

$$\varphi = \sum_{\substack{A \cup B = \{1,\cdots,n\} \\ A = \{\ell_1,\cdots,\ell_r\} \\ n \notin B}} \prod_{s \in B} 2^{-\epsilon(i_{s+1} - i_s)} \sum_{j_1 \in J_{\ell_1}} \cdots \sum_{j_r \in J_{\ell_r}} \partial_{j_1} \cdots \partial_{j_r} [\varphi_{A,B,j_1,\cdots,j_r}]$$

Here the outer sum is taken over all decompositions $\{1,\cdots,n\}$ into two disjoint subsets A and B such that $n \in A$.

Remark If φ has weak cancellation relative to $I \in \mathcal{E}_n$ in $\mathbb{R}^N$ then φ has weak cancellation relative to $I_B \in \mathcal{E}_s$ in $\mathbb{R}^{a_{m_1}} \oplus \cdots \oplus \mathbb{R}^{a_{m_s}} \subset \mathbb{R}^N$, $s \le n$

It follows that for any $B \subset \{1,\cdots,n\}$,

$$\varphi = \sum_{\substack{B_1 \cup B_2 = B \\ B_1 \cap B_2 = \phi, \\ n \notin B_2 \text{ if } n \in B}} \prod_{j \in B_2} 2^{-\epsilon(i_{j+1} - i_j)} \left(\prod_{k \in B_1} \partial_{\sigma(k)} \right) \Phi_{B_1,B_2,\sigma}$$

where $\sigma : B_2 \longrightarrow \{1,\cdots,N\}$ so that $\sigma(k) \in J_k = \{p_k, p_k+1, \cdots, q_k\}$, and each $\Phi_{B_1,B_2,\sigma} \in \mathcal{S}(\mathbb{R}^N)$ is normalized in terms of φ

In particular, if $B = \{1\}$, then, $\exists\ \varphi_j\ (j \in J_1)$, $\varphi_0 \in \mathcal{S}$, normalized relative to φ, ∋

$$\varphi = \sum_{j \in J_1} \partial_j [\varphi_j] + 2^{-\epsilon(i_2 - i_1)} \varphi_0$$

<u>Prop.</u> (Nagel - Ricci - Stein - Wainger, 2012)

Let $\varepsilon > 0$ and $I \in \mathcal{E}_n$. A function $\psi \in C_c^\infty(\mathbb{R}^N)$ (resp. $\psi \in \mathcal{S}(\mathbb{R}^N)$) has weak cancellation with parameter ε and multi-index I if and only if for every decomposition $\{1, \cdots, n\} = A \cup B$ with disjoint subsets A and B, there exists $\psi_A \in C_c^\infty(\mathbb{R}^{N_a})$ (resp. $\psi_A \in \mathcal{S}(\mathbb{R}^{N_a})$) normalized in terms of ψ ∍

$$\int_{\mathbb{R}^{N_b}} \psi(x_A, x_B)\, dx_B = \begin{cases} 0 & \text{if } n \in B \\ \prod_{k \in B} 2^{-\varepsilon(i_{k+1} - i_k)} \psi_A(x_A) & \text{if } n \notin B \end{cases}$$

<u>Dyadic decompositions of flag kernels</u>

<u>Thm</u> (Nagel - Ricci - Stein - Wainger, 2012)

Every flag kernel $\mathcal{K}$ relative to the flag $\mathcal{F}$ has a decomposition

$$\mathcal{K} = \mathcal{K}_0 + \sum_{j=1}^{n} \mathcal{K}_j$$

with the following properties that

(i) For each $I \in \mathcal{E}_n$, there is a function $\varphi^I \in C_c^\infty(\mathbb{R}^N)$ supported in the unit ball B_1 with strong cancellation and uniformly bounded seminorms $\|\varphi\|_{(m)}$ ∍ the series

$$\mathcal{K}_0 = \sum_{I \in \mathcal{E}_n} [\varphi^I]_I$$

converges in the sense of distributions. Moreover, for each $I \in \mathcal{E}_n$, $\varphi^I(x_1, \cdots, x_n) = 0$ for $|x_1| \leq \frac{1}{8}$.

Here, $[f]_I(x) = 2^{-\sum_{l=1}^{n} Q_l i_l} f(2^{-i_1} \cdot x_1, \cdots, 2^{-i_n} \cdot x_n)$.

(ii) For each $1 \leq j \leq n$, each $\mathcal{K}_j$ is a flag kernel relative to a flag strictly coarser than $\mathcal{F}$.

Pf of Thm

1^{st}. By the characterization of the flag kernels in terms of their Fourier transform one has that

$$\mathcal{K} = \mathcal{K}_0 + \sum_{j=1}^{n} \mathcal{K}_j,$$

where the series

$$\mathcal{K}_0 = \sum_{I \in \mathcal{E}_n} [\Psi^I]_I \qquad \text{with } \Psi^I \in \mathcal{S}(\mathbb{R}^N)$$

converges in the sense of distributions, and for $1 \le j \le n$, $\mathcal{K}_j$ is a flag kernel relative to a flag coarser than $\mathcal{F}$.

2^{nd}. According to the decomposition of the test functions in $\mathcal{S}(\mathbb{R}^N)$, there exist functions $\psi^{k,I} \in C_c^\infty(\mathbb{R}^N)$ supported in the unit ball with strong cancellation such that

$$\psi^I(x) = \sum_{k=0}^{\infty} 2^{-k(Q_1 + \cdots + Q_n)} \psi^{k,I}(2^{-k} \cdot x)$$

Moreover, for every $\delta > 0$ and every $m \in \mathbb{N}$, there exists $s_m \in \mathbb{Z}$ such that

$$\|\psi^{k,I}\|_{(m)} \le 2^{-k\delta} \|\Psi^I\|_{m+s_m}$$

So, the series

$$\sum_{k=0}^{\infty} \psi^{k,I} := \tilde{\varphi}^I$$

converges in $C_c^\infty(\mathbb{R}^N)$ to a function with strong cancellation, supported in the unit ball. Thus,

$$\mathcal{K}_0 = \sum_{I \in \mathcal{E}_n} [\tilde{\varphi}^I]_I$$

converges in the sense of distributions.

3^{rd}. By the decomposition of test functions in $C_c^\infty(\mathbb{R}^N)$, it follows that for each $I \in \mathcal{E}_n$, there exist functions $\varphi^{j,I} \in C_c^\infty(\mathbb{R}^N)$ with strong cancellation, supported in the unit ball, normalized in terms of $\tilde{\varphi}^I$, and vanishing when $|x_1| \le \frac{1}{8}$ such that

$$\tilde{\varphi}^I(x_1, \cdots, x_n) = \sum_{j=-\infty}^{0} 2^{-jQ_1} \varphi^{j,I}(2^{-j} \cdot x_1, x_2, \cdots, x_n)$$

Then we have

$$\sum_{I \in \mathcal{E}_n} [\tilde{\varphi}^I]_I (x) = \sum_{I \in \mathcal{E}_n} \Big[\sum_{j=-\infty}^{0} \varphi^{j,I} \Big]_I (x)$$

and the series

$$\sum_{j=-\infty}^{0} \varphi^{j,I} =: \varphi^I$$

converges in $C_c^\infty(\mathbb{R}^N)$ with strong cancellation, supported in the unit ball, and vanishing when $|x_1| \le \frac{1}{8}$. Thus,

$$\mathcal{K}_0 = \sum_{I \in \mathcal{E}_n} [\varphi^I]_I$$

converges in the sense of distributions. □.

Dyadic sums with weak cancellation

Thm For each $I \in \mathcal{E}_n$, let $\varphi^I \in \mathcal{S}(\mathbb{R}^N)$ have uniformly bounded seminorms $\|\varphi\|_M$ for each $M \in \mathbb{N}_0$ and have weak cancellation w.r.t. I with some parameter $\varepsilon > 0$.

Then,

(i) If $F \subset \mathcal{E}_n$ is any finite set, then the Schwartz function

$$\mathcal{K}_F = \sum_{I \in F} [\varphi^I]_I$$

defines a flag kernel relative to the flag $\mathcal{F}$ with bounds independent of the set F.

(ii) Let $\{F_j\}_j$ be any increasing sequence of finite subsets of $\mathcal{E}_n$ with $\mathcal{E}_n = \bigcup_{j=1}^{\infty} F_j$. Then, for any $\psi \in \mathcal{S}(\mathbb{R}^N)$, the limits

$$\lim_{j\to\infty} \langle K_{F_j}, \psi \rangle = \lim_{j \to \infty} \sum_{I \in F_j} \int_{\mathbb{R}^N} [\varphi^I]_I(x)\, \psi(x)\, dx$$

exists and defines a flag kernel $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ and we write the limit as

$$\mathcal{K} = \lim_{F \nearrow \mathcal{E}_n} \sum_{I \in F} [\varphi^I]_I$$

Pf of the Thm

(i) Size estimates: For each $I \in \mathcal{E}_n$ and $\varphi^I \in \mathcal{S}(\mathbb{R}^N)$, since

$$\partial_x^\alpha\left([\varphi^I]_I\right)(x) = 2^{-\sum_{\ell=1}^{n} i_\ell [\![\bar{\alpha}_\ell]\!]} [\partial_x^\alpha \varphi^I]_I(x) \qquad \forall \alpha \in \mathbb{N}_0^N$$

one has that for any $M > \sum_{\ell=1}^{n} i_\ell \left([\![\bar{\alpha}_\ell]\!] + Q_\ell\right)$, there exists a constant $C = C(\alpha, M$ $\ni$

$$|\partial_x^\alpha K_F(x)| = \left|\sum_{I\in F} \partial_x^\alpha\left([\varphi^I]_I\right)(x)\right|$$

$$= \left|\sum_{I\in F} 2^{-\sum_{\ell=1}^{n} i_\ell ([\![\bar{\alpha}_\ell]\!] + Q_\ell)} (\partial_x^\alpha \varphi^I)(2^{-i_1}\cdot x_1, \dots, 2^{-i_n}\cdot x_n)\right|$$

$$\le C \sum_{I\in F} 2^{-\sum_{\ell=1}^{n} i_\ell ([\![\bar{\alpha}_\ell]\!] + Q_\ell)} \left(1 + \sum_{k=1}^{n} 2^{-i_k} N_k(x_k)\right)^{-M}$$

$$\le C \prod_{k=1}^{n} \left(N_1(x_1) + \dots + N_k(x_k)\right)^{-Q_k - [\![\bar{\alpha}_k]\!]}$$

Cancellation conditions We need to show that for any $R = (R_1, \dots, R_s) \in (\mathbb{R}_+)^s$ and for any $\psi \in C_c^\infty(\mathbb{R}^{N_b})$ the integrals of the form

$$K^{\#}_{F,\psi} := \int_{\mathbb{R}^{N_b}} K_F(x_A, x_B)\, \psi(R\cdot x_B)\, dx_B$$

s.t. the size estimates of part (i). Here,

$$A = \{\ell_1, \dots, \ell_r\}, \qquad B = \{m_1, \dots, m_s\} = \{1, \dots, n\} \setminus A$$

and

$$x_A = (x_{\ell_1}, \dots, x_{\ell_r}) \in \mathbb{R}^{N_a}, \qquad N_a = a_{\ell_1} + \dots + a_{\ell_s},$$

$$x_B = (x_{m_1}, \dots, x_{m_s}) \in \mathbb{R}^{N_b}, \qquad N_b = a_{m_1} + \dots + a_{m_s}.$$

For $I = (i_1, \dots, i_n) \in F \subset \mathcal{E}_n$, set

$$E_1 = \{I_A = (i_{\ell_1}, \dots, i_{\ell_r}) \in \mathcal{E}_r \mid (i_1, \dots, i_n) \in F\}$$

and for $I_A \in E_1$, set

$$E_2(I_A) = \{I_B = (i_{m_1}, \dots, i_{m_s}) \in \mathcal{E}_s \mid (i_1, \dots, i_n) \in F\}$$

then we write $I = (I_A, I_B)$ with $I_A \in E_1$ and $I_B \in E_2(I_A)$. Thus,

$$K^{\#}_{F,\psi} = \sum_{I \in F \subset \mathcal{E}_n} \int_{\mathbb{R}^{N_b}} [\varphi^I]_I (x_A, x_B)\, \psi(R \cdot x_B)\, dx_B$$

$$= \sum_{\substack{I=(I_A, I_B) \\ \in F \subset \mathcal{E}_n}} \int_{\mathbb{R}^{N_b}} [\varphi^I]_{I_A} (x_A, x_B)\, \psi(R \cdot x_B)\, dx_B$$

$$= \sum_{I_A \in E_1} \Big[\sum_{I_B \in E_2(I_A)} \Theta^{(I_A, I_B)} \Big]_{I_A} (x_A),$$

where

$$\Theta^I(x_A) = \Theta^{(I_A, I_B)}(x_A) = \int_{\mathbb{R}^{N_b}} \varphi^I(x_A, x_B)\, \psi(R \cdot x_B)\, dx_B \in \mathcal{S}(\mathbb{R}^{N_a})$$

is normalized relative to φ^I.

Since $\varphi^I \in \mathcal{S}(\mathbb{R}^N)$ has weak cancellation

$\rightsquigarrow$ φ^I can be written as a sum of terms of form

$$\prod_{j \in B_2} 2^{-\epsilon(i_{j+1} - i_j)} \Big(\prod_{k \in B_1} \partial_{\sigma(k)} \Big) \tilde{\varphi}^I_{B_1, B_2, \sigma}$$

where $B_1 \cup B_2 = B$ with $B_1 \cap B_2 = \phi$ and $n \in B_1$ if $n \in B$, and $\sigma : B_1 \to \{1, \cdots, N\}$ so that $\sigma(k) \in J_k = \{p_k, p_k + 1, \cdots, q_k\} \subset \{1, \cdots, N\}$, and each $\tilde{\varphi}^I_{B_1, B_2, \sigma} \in \mathcal{S}(\mathbb{R}^N)$ is normalized in terms of φ^I.

$\rightsquigarrow$ $\Theta^I = \Theta^{(I_A, I_B)}$ is a finite sum of terms of the form

$$\prod_{j \in B_2} 2^{-\epsilon(i_{j+1} - i_j)} \int_{\mathbb{R}^{N_b}} \Big(\prod_{k \in B_1} \partial_{\sigma(k)} \Big) \Big[\tilde{\varphi}^{(I_A, I_B)}_{B_1, B_2, \sigma} \Big] \psi(R \cdot x_B)\, dx_B$$

$\because \psi \in C^\infty_c(\mathbb{R}^N)$ $\Longrightarrow$ $\sum_{I_B \in E_2(I_A)} \Theta^{(I_A, I_B)}$ converges to a normalized Schwartz functions

$\xRightarrow[\text{of part (i)}]{\text{by size estimates}}$ $\sum_{I_A \in E_1} \Big[\sum_{I_B \in E_2(I_A)} \Theta^{(I_A, I_B)} \Big]_{I_A}$ s.t. the size estimates of part (i)

i.e. $K^{\#}_{F,\psi}$ s.t. size estimates of part (i). □

(ii) For each $I \in \mathcal{E}_n$, since $\varphi^I \in \mathcal{S}(\mathbb{R}^N)$ has weak cancellation relative to I, there exist functions φ^I_ℓ ($\ell \in J_1$), φ^I_0 in $\mathcal{S}(\mathbb{R}^N)$ normalized relative to φ^I so that

$$\varphi^I = \sum_{\ell \in J_1} \partial_\ell (\varphi^I_\ell) + 2^{-\epsilon(i_2 - i_1)} \varphi^I_0,$$

thus,

$$[\varphi^I]_I = \sum_{\ell \in J_1} 2^{i_1 d_\ell} \partial_\ell \left([\varphi^I_\ell] \right) + 2^{-\epsilon(i_2 - i_1)} [\varphi^I_0]_I .$$

Then if $F \subset \mathcal{E}_n$ is a finite subset, and $K_F(x) = \sum_{I \in F} [\varphi^I]_I (x)$, we have by integration by parts that

$$\int_{\mathbb{R}^N} K_F(x) \psi(x)\, dx = - \sum_{\ell \in J_1} \int_{\mathbb{R}^N} K_F^\ell(x) \partial_\ell \psi(x)\, dx + \int_{\mathbb{R}^N} K_F^0(x) \psi(x)\, dx$$

where

$$K_F^\ell(x) = \sum_{I \in F} 2^{i_1 d_\ell} [\varphi^I_\ell]_I (x), \qquad \text{for } \ell \in J_1$$

&

$$K_F^0(x) = \sum_{I \in F} 2^{-\epsilon(i_2 - i_1)} [\varphi^I_0]_I (x).$$

Then for $\alpha = (\bar{\alpha}_1, \dots, \bar{\alpha}_n) \in \mathbb{N}_0^N$ we have for $\ell \in J_1$ and $M > \sum_{\ell=1}^n i_\ell (\llbracket \bar{\alpha}_\ell \rrbracket + Q_\ell)$

$$\left| \partial_x^\alpha K_F^\ell(x) \right| = \left| \sum_{I \in F} 2^{i_1 d_\ell} 2^{-\sum_{j=1}^n i_j (Q_j + \llbracket \bar{\alpha}_j \rrbracket)} \partial_x^\alpha (\varphi^I_\ell)(2^{-i_1} \cdot x_1, \dots, 2^{-i_n} \cdot x_n) \right|$$

$$\le \sum_{I \in F} 2^{i_1 d_\ell} 2^{-\sum_{j=1}^n i_j (Q_j + \llbracket \bar{\alpha}_j \rrbracket)} \left(1 + \sum_{k=1}^n 2^{-i_k} N_k(x_k) \right)^{-M}$$

$$\le C \underbrace{N_1(x_1)^{d_\ell} \prod_{j=1}^n \left(N_1(x_1) + \dots + N_j(x_j) \right)^{-Q_j - \llbracket \bar{\alpha}_j \rrbracket}}_{\in L^1(\mathbb{R}^N)}$$

and

$$\left| \partial_x^\alpha K_F^0(x) \right| = \left| \sum_{I \in F} 2^{-\epsilon i_2 + \epsilon i_1} 2^{-\sum_{j=1}^n i_j (Q_j + \llbracket \bar{\alpha}_j \rrbracket)} (\partial_x^\alpha \varphi^I_0)(2^{-i_1} \cdot x_1, \dots, 2^{-i_n} \cdot x_n) \right|$$

$$\le \sum_{I \in F} 2^{-\epsilon i_2 + \epsilon i_1} 2^{-\sum_{j=1}^n i_j (Q_j + \llbracket \bar{\alpha}_j \rrbracket)} \left(1 + \sum_{k=1}^n 2^{-i_k} N_k(x_k) \right)^{-M}$$

$$\le C \underbrace{N_1(x_1)^{\epsilon} \left(N_1(x_1) + N_2(x_2) \right)^{-\epsilon} \prod_{j=1}^n \left(N_1(x_1) + \dots + N_j(x_j) \right)^{-Q_j - \llbracket \bar{\alpha}_j \rrbracket}}_{\in L^1(\mathbb{R}^N)}$$

Then by dominated convergence thm, part (ii) follows and

$$\lim_{F \nearrow \mathcal{E}_n} \langle \mathcal{K}_F, \psi \rangle = -\sum_{\ell \in J_1} \int_{\mathbb{R}^N} \mathcal{K}^{\ell}(x)\, \partial_\ell \psi(x)\, dx + \int_{\mathbb{R}^N} \mathcal{K}^{0}(x)\, \psi(x)\, dx$$

where

$$\mathcal{K}^{\ell}(x) = \sum_{I \in \mathcal{E}_n} 2^{i_1 d_\ell} [\varphi^I_\ell]_I (x) \qquad \text{for } \ell \in J_1$$

&

$$\mathcal{K}^{0}(x) = \sum_{I \in \mathcal{E}_n} 2^{-\epsilon(i_2 - i_1)} [\varphi^I_0]_I (x) \qquad \square$$

Invariance of flag kernels under change of variables

Thm (Nagel-Ricci-Stein-Wainger, 2012)

Let $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ be a flag kernel relative to the flag $\mathcal{F}$ associated to the decomposition $\mathbb{R}^N = \mathbb{R}^{a_1} \oplus \dots \oplus \mathbb{R}^{a_n}$. If $y = F(x)$ is an admissible change of variables, then $\mathcal{K} \circ F$ is a flag kernel for the same decomposition.

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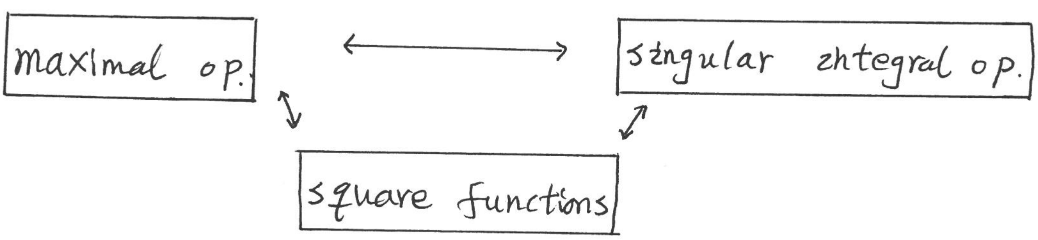
3. L^p boundedness of the op. $T_{\mathcal{K}} : f \mapsto f * \mathcal{K}$ on G for $1 < p < \infty$.

Thm (Nagel-Ricci-Stein-Wainger, 2012)

The op. $T_{\mathcal{K}} : f \mapsto f * \mathcal{K}$ with flag kernel $\mathcal{K}$ is bounded on $L^p(G)$ for $1 < p < \infty$.

To show this theorem, we first recall some related classical theory of harmonic analysis in one-parameter case.



① Hardy-Littlewood maximal op. $\mathcal{M}$: $\forall f \in L_{loc}(\mathbb{R}^N)$

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{m(B(0,r))} \int_{B(0,r)} |f(x-y)|\, dy \qquad B: \text{ball}$$

$$\approx \sup_{r>0} \frac{1}{m(Q(0,r))} \int_{Q(0,r)} |f(x-y)|\, dy \qquad Q: \text{cube}$$

Property (i) $\mathcal{M}$ is of weak (1, 1) type and is of (p, p) type for $1<p<\infty$

↑ by Vitali covering lemma (requires: ∃quasi metric, i.e. doubling property)

(ii) $\left\| \left[\sum_j (\mathcal{M}(f_j))^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^N)} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^N)}$ for $1<p<\infty$

(iii) $\mathcal{M}$ is invariant with one-parameter dilation $\lambda \cdot \boldsymbol{x} = (\lambda x_1, \cdots, \lambda x_N)$ for any $\lambda > 0$

② Poisson kernel : $P_t(x) = t^{-N} \left(1 + \frac{|x|^2}{t^2}\right)^{-\frac{N+1}{2}}$

Property $\sup_{t>0} |f * P_t(x)| \leq \mathcal{M}(f)(x)$

③ Smooth Calderón-Zygmund kernel and singular integral operators :

Let $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^N)$ is a smooth Calderón-Zygmund kernel, which coincides with a function K when away from 0, and op. $T: f \mapsto f * \mathcal{K}$.

Suppose $\varphi \in C_c^\infty(\mathbb{R}^N)$ is supported on the unit ball of $\mathbb{R}^N$ and $\int_{\mathbb{R}^N} \varphi = 0$.

Set $\varphi_t(x) := t^{-N} \varphi(t^{-1} \cdot x)$, $t^{-1} \cdot \boldsymbol{x} = (t^{-1} x_1, \cdots, t^{-1} x_N)$. Then,

(i) $\sup_{t>0} |K * \varphi_t(x)| \lesssim P_t(x)$

(ii) $|(Tf * \varphi_t)(x)| \lesssim (\mathcal{M}f)(x)$

(iii) $|\varphi_s * (Tf) * \varphi_t(x)| \lesssim \gamma(s,t)\, (\mathcal{M}f)(x)$,

where $\gamma(s,t) = \min\left(\frac{s}{t}, \frac{t}{s}\right)^\delta$ for some $\delta > 0$.

④ <u>Square function</u> Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ have support in the unit ball of $\mathbb{R}^N$, and $\psi \in \mathcal{S}(\mathbb{R}^N)$, ∋

$$\int_0^\infty \hat{\varphi}(t\xi)\,\hat{\psi}(t\xi)\,\frac{dt}{t} = 1. \qquad (*)$$

Define square functions $S_\varphi f$ and $S_\psi f$ as

$$S_\varphi(f)(x) = \left(\int_0^\infty |(f * \varphi_t)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int_0^\infty |(f * \psi_t)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where $\varphi_t(x) = t^{-N}\varphi(t^{-1}\cdot x)$ and $\psi_t(x) = t^{-N}\psi(t^{-1}\cdot x)$ for $t>0$ and $x \in \mathbb{R}^N$.

Then, for any $1<p<\infty$,

$$\|f\|_{L^p(\mathbb{R}^N)} \approx \|S_\varphi(f)\|_{L^p(\mathbb{R}^N)} \approx \|S_\psi(f)\|_{L^p(\mathbb{R}^N)} \qquad (**)$$

<u>Remark</u> We add condition $(*)$ only to get that

$$f = \int_0^\infty \varphi_t * \psi_t * f \,\frac{dt}{t}$$

When without condition $(*)$, we still have $(**)$.

Now we use the above close relationship among Hardy-Littlewood maximal op., Poisson kernel, singular integral op. with smooth Calderón-Zygmund convolution kernel, and square function to obtain the L^p boundedness of the op. $T: f \mapsto f * \mathcal{K}$ in Euclidean space $\mathbb{R}^N$ for $1<p<\infty$ with smooth Calderón-Zygmund kernel.

Here we will not use the translation invariant structure of this op. here.

<u>Show</u> Let $\mathcal{K}$ be a smooth Calderón-Zygmund kernel in $\mathbb{R}^N$. Then op. $T: f \mapsto f * \mathcal{K}$ is bounded on $L^p(\mathbb{R}^N)$ for $1<p<\infty$.

<u>Pf</u> 1st. $\|Tf\|_{L^p(\mathbb{R}^N)} \lesssim \|S_\varphi(Tf)\|_{L^p(\mathbb{R}^N)}$

2nd. $$\left(S_\varphi(Tf)(x)\right)^2 \equiv \int_0^\infty |(Tf) * \varphi_t(x)|^2 \frac{dt}{t}$$

Since $\int_0^\infty \hat{\varphi}(t\xi)\,\hat{\psi}(t\xi)\frac{dt}{t} = 1$ we have $f = \int_0^\infty (\varphi_s * \psi_s * f)\frac{ds}{s}$,

thus,

$$Tf = f * K = \int_0^\infty (\varphi_s * \psi_s * f * K)\frac{ds}{s}$$

&

$$\varphi_t * (Tf) = \int_0^\infty (\varphi_t * \varphi_s * \psi_s * f * K)\frac{ds}{s}$$

$$= \int_0^\infty \varphi_t * T(\varphi_s * \psi_s * f)\frac{ds}{s}$$

Since

$$|\varphi_t * T(\varphi_s * \psi_s * f)(x)| \lesssim \gamma(s,t)\,\mathcal{M}(\psi_s * f)(x)$$

we have

$$|\varphi_t * (Tf)(x)|^2 \lesssim \left(\int_0^\infty \gamma(s,t)\,\mathcal{M}(\psi_s * f)(x)\frac{ds}{s}\right)^2$$

$$\lesssim \int_0^\infty \gamma(s,t)\left(\mathcal{M}(\psi_s * f)(x)\right)^2\frac{ds}{s}$$

$$\times \int_0^\infty \gamma(s,t)\frac{ds}{s} \qquad \text{by Cauchy-Schwartz ineq.}$$

Note that $\sup_{t>0}\int_0^\infty \gamma(s,t)\frac{ds}{s} \le A < \infty$. Thus,

$$\left(S_\varphi(Tf)(x)\right)^2 \lesssim \int_0^\infty\int_0^\infty \gamma(s,t)\left(\mathcal{M}(\psi_s * f)(x)\right)^2\frac{ds}{s}\frac{dt}{t}$$

$$\le \left(\sup_{s>0}\int_0^\infty \gamma(s,t)\frac{dt}{t}\right)\left[\int_0^\infty \left(\mathcal{M}(\psi_s * f)(x)\right)^2\frac{ds}{s}\right]$$

$$\rightsquigarrow S_\varphi(Tf)(x) \lesssim \left[\int_0^\infty \left(\mathcal{M}(\psi_s * f)(x)\right)^2\frac{ds}{s}\right]^{\frac{1}{2}} =: S_\psi^\#(f)(x).$$

3rd. Since Hardy-Littlewood maximal op. $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^N, \ell^2)$ for $1<p<\infty$ one gets that for any measurable function $F_t(x)$ of $(t,x)\in\mathbb{R}_+\times\mathbb{R}^N$ that

$$\left\|\left[\int_0^\infty \left(\mathcal{M}(F_t)(x)\right)^2 dt\right]^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^N)} \lesssim \left\|\left(\int_0^\infty |F_t(x)|^2 dt\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^N)}$$

Thus,

$$\|S_\psi^\#(f)\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}, \quad 1<p<\infty. \qquad \square$$

Now show: op. $T: f \mapsto f * \mathcal{K}$ with flag kernel $\mathcal{K}$ is bounded on $L^p(G)$, $1<p<\infty$.

Pf Notice that for any $x, y \in G = \mathbb{R}^N$, the k^{th} component of $x \cdot y$ is

$$(x\cdot y)_k = x_k + y_k + P_k(x,y) = x_k + y_k + \sum_{\alpha,\beta\in\Lambda_k} C_k^{\alpha,\beta} x_1^{\alpha_1}\cdots x_{k-1}^{\alpha_{k-1}} y_1^{\beta_1}\cdots y_{k-1}^{\beta_{k-1}}$$

where

$$\Lambda_k = \left\{(\alpha;\beta) = (\alpha_1,\cdots,\alpha_{k-1};\beta_1,\cdots,\beta_{k-1}) \,\middle|\, \sum_{\ell=1}^{k-1} d_\ell(\alpha_\ell+\beta_\ell) = d_k\right\},\ 1\le k\le N$$

We known that op. T under such group multiplication is not translation invariant, but left-invariant on G. Nagel, Ricci, Stein, and Wainger's idea is to use its relationships with maximal op. and square functions.

Since the flag kernel is invariant under the change of coordinate, Nagel-Ricci-Stein-Wainger do calculations in a particular coordinate system so that $x \in G = \mathbb{R}^N$ can be written as a product

$$x = x' \cdot x'' \quad \text{with} \quad x' = (x_1,\cdots,x_{k-1},0,\cdots,0) \quad \text{and} \quad x'' \in G_k$$

Here

$$G_k = \{x = (x_1,\cdots,x_n)\in\mathbb{R}^N \mid x_1 = \cdots = x_{k-1} = 0\}$$
$$= \{(0,\cdots,0,x_k,\cdots,x_n) \mid x_\ell \in \mathbb{R}^{a_\ell},\ k\le\ell\le n\}, \qquad 1\le k\le n.$$

Then, G_k is a subgroup of G and $G = G_1 \supset G_2 \supset \cdots \supset G_n$

$\rightsquigarrow \forall x \in G_k$,

$$x = (x_k, x_{k+1},\cdots,x_n) = (x_k,0,\cdots,0)\cdot(0,x_{k+1},\cdots,x_n)$$
$$= (x_k)\cdot x' \quad \text{with } x' \in G_{k+1}$$

We first have a look what the corresponding maximal op. and square functions could be on G.

① Maximal op.

(i) $n=1$: $$\mathcal{M} f(x) = \sup_{r>0} \frac{1}{m(B(0,r))} \int_{B(0,r)} |f(x-y)|\,dy$$

$$\approx \sup_{r>0} \frac{1}{m(Q(0,r))} \int_{Q(0,r)} |f(x-y)|\,dy, \quad Q: \text{cube}$$

$\mathcal{M}$ is invariant with one-parameter dilation

(ii) $n \geq 2$: The maximal op. should be sup of average over rectangles.

For $s = (s_k, \cdots, s_n)$, let

$$R_s^{(k)} := \{ (x_k, \cdots, x_n) \in G_k : |x_k| \leq s_k^k, \cdots, |x_n| \leq s_n^n \}$$

$$R_s := R_s^{(1)} \quad \text{for } s = (s_1, \cdots, s_n)$$

We say that the size of the rectangle $R_s^{(k)}$ is acceptable if $s_k \leq s_{k+1} \leq \cdots \leq s_n$

We let $m(E)$ denote the Lebesgue measure of a set $E \subseteq G = G_1$

$m_k(E)$ of a set $E \subseteq G_k$

Def The strong maximal op. $\mathcal{M}$ defined on $G = G_1$ is given by

$$\mathcal{M}(f)(x) = \sup \frac{1}{m(R_s)} \int_{R_s} |f(x \cdot y^{-1})|\,dy$$

where the supremum is taken over all acceptable rectangles $R_s = R_s^{(1)} \subseteq G = G_1$.

Thm 3.1 (i) $\mathcal{M}$ is bounded on $L^p(G)$ for $1<p<\infty$

(ii) $$\left\| \left\{ \sum_j [\mathcal{M}(f_j)]^2 \right\}^{\frac{1}{2}} \right\|_{L^p(G)} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \quad \text{for } 1<p<\infty$$

(iii) $$\left\| \left[\int_{(\mathbb{R}_+)^n} \left(\mathcal{M}(F_{t_1,\cdots,t_n})(\cdot) \right)^2 dt_1 \cdots dt_n \right]^{\frac{1}{2}} \right\|_{L^p(G)} \lesssim \left\| \left(\int_{(\mathbb{R}_+)^n} |F_{t_1,\cdots,t_n}(\cdot)|^2 dt_1 \cdots dt_n \right)^{\frac{1}{2}} \right\|_{L^p(G)}$$

for any measurable function $F_{t_1,\cdots,t_n}(x)$ of $(t_1,\cdots,t_n,x) \in (\mathbb{R}_+)^n \times G$.

We consider a maximal op. $\mathcal{M}_k$ on G_k : $\forall f \in L^1_{loc}(G_k)$,

$$\mathcal{M}_k(f)(x) = \sup_{\rho>0} \frac{1}{m_k(B_\rho^{(k)})} \int_{B_\rho^{(k)}} |f(x\cdot y^{-1})|\, dy \quad \text{for } x \in G_k,\ 1 \le k \le n$$

where m_k is the Lebesgue measure on G_k, and $B_\rho^{(k)}$ is the automorphic one-parameter ball given by

$$B_\rho^{(k)} = \left\{ (x_k, \cdots, x_n) \in G_k \;\middle|\; |x_k| < \rho^k,\ |x_{k+1}| < \rho^{k+1}, \cdots, |x_n| < \rho^n \right\}$$

Note that $B_\rho^{(k)}$ s.t. the required properties for both the Vitali covering argument and Calderón-Zygmund decomposition argument.

<u>Property for $\mathcal{M}_k$</u>

(i) $\mathcal{M}_k$ is of weak (1, 1) type and is of (p, p) type on G, $1<p<\infty$.

(ii) $\left\| \left[\sum_j (\mathcal{M}_k(f_j))^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)}$, $1<p<\infty$

<u>Pf of (ii)</u> ① $p=2$: ✓

② $p=1$: using Calderón-Zygmund decomposition one proves that $\mathcal{M}_k$ is bounded from $L^1(G_k, \ell^2)$ to $L^{1,\infty}(G_k, \ell^2)$

③ $1<p<2$: by Marczhkiewicz interpolation

④ $p>2$: by a weighted norm ineq

$$\int_{G_k} (\mathcal{M}_k(f)(x))^2 w(x)\, dx \lesssim \int_{G_k} |f(x)|^2 \mathcal{M}_k w(x)\, dx.$$

Here, w is any positive function.

Lifting Suppose op. $L : f \longmapsto f * u$ is a solution op. on G_k with $f \in \mathcal{S}(G_k)$, $u \in \mathcal{S}'(G_k)$.

Then,

$$L(f)(x) = \int_{G_k} f(x y^{-1})\, u(y)\, dy := \langle u, F_x \rangle,$$

where

$$F_x(y) := f(x y^{-1}) \quad \text{for } y \in G_k$$

Then, $F_x \in \mathcal{S}(G_k)$

We can lift L to a convolution op. $\tilde{L}$ on G:

$$\tilde{L}(f)(x) = \int_{G_k} f(x y^{-1})\, u(y)\, dy = \langle u, F_x \rangle \quad \forall f \in \mathcal{S}(G),\ \forall x \in G.$$

Here, we choose a particular coordinate system, $x = (x_1, \cdots, x_n) \in G$ can be written as a product

$$x = x' \cdot x'' \quad \text{with } x' = (x_1, \cdots, x_{k-1}, 0, \cdots, 0) \text{ and } x'' \in G_k$$

With this coordinate system, we define $\tilde{u} \in \mathcal{S}'(G)$ as

$$\tilde{u} = \delta_{x'} \otimes u$$

and set

$$\tilde{L}(f)(x) := (f * \tilde{u})(x) \quad \text{for } f \in \mathcal{S}(G) \text{ and } x \in G$$

Then,

$$\tilde{L}(f)(x) = \int_G f(x \cdot y^{-1})\, \tilde{u}(y)\, dy = \int_G f(x \cdot (y', y'')^{-1}) \left(\delta_{y'} \otimes u(y'')\right) dy$$

$$x \cdot y^{-1} = x \cdot (y' \cdot y'')^{-1}$$

$$= \int_{G_k} f(x \cdot y''^{-1})\, u(y'')\, dy'' = \int_{G_k} f((x' \cdot x'') \cdot y''^{-1})\, u(y'')\, dy''$$

$$= \int_{G_k} f^{x'}(x'' \cdot y^{-1})\, u(y)\, dy$$

$$= T(f^{x'})(x''), \qquad f^{x'}(y) = f(x' \cdot y) \text{ for } y \in G_k$$

Claim

If op. $\mathcal{T}$ is bounded on $L^p(G_k)$, then, $\tilde{\mathcal{T}}$ is bounded on $L^p(G)$, $1<p<\infty$.

Pf $\mathcal{T}$ is bounded on $L^p(G_k)$

$$\Rightarrow \quad \| T(f^{x'}) \|_{L^p(G_k)} \lesssim \| f^{x'} \|_{L^p(G_k)}$$

$$\| T(f^{x'}) \|_{L^p(G_k)} = \| \tilde{T}(f) \|_{L^p(G_k)}, \qquad \| f^{x'} \|_{L^p(G_k)} = \| f(x', \cdot) \|_{L^p(G_k)}$$

integration in x'

$$\Rightarrow \quad \| \tilde{T}(f) \|_{L^p(G)} \lesssim \| f \|_{L^p(G)}.$$

• $$\mathcal{M}(f)(x) := \sup_{R_s} \frac{1}{m(R_s)} \int_{R_s} |f(x \cdot y^{-1})| \, dy = \sup_{R_s} |f| * \mathcal{I}_{R_s}(x),$$

$$\mathcal{M}_k(f)(x) := \sup_{\rho>0} \frac{1}{m_k(B_\rho^{(k)})} \int_{B_\rho^{(k)}} |f(x \cdot y^{-1})| \, dy = \sup_{\rho>0} |f| * \mathcal{I}_{B_\rho^{(k)}}(x)$$

where $\mathcal{I}_{R_s}(x) = \mathbb{1}_{R_s}(x) \,/\, m(R_s)$,

$$\mathcal{I}_{B_\rho^{(k)}}(x) = \mathbb{1}_{B_\rho^{(k)}}(x) \,/\, m_k(B_\rho^{(k)})$$

$$\rightsquigarrow \quad \tilde{\mathcal{M}}_k(f)(x) = \sup_{\rho>0} |f| * \left(\delta_{x_1, \dots, x_{k-1}} \otimes \mathcal{I}_{B_\rho^{(k)}} \right)(x), \qquad k \geq 2$$

$\rightsquigarrow \quad \tilde{\mathcal{M}}_k$ is bounded on $L^p(G)$, $1<p<\infty$.

and
$$\left\| \left[\sum_j \left(\tilde{\mathcal{M}}_k(f_j) \right)^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \quad \forall k. \ (1 \leq k \leq n)$$

Observation $\quad \mathbb{1}_{R^{(k)}_{cS}} \lesssim \mathbb{1}_{B^{(k)}_{S_k}} * \left(\delta_{x_k} \otimes \mathbb{1}_{R^{(k+1)}_{\bar{S}}}\right) \quad (\#1)$

whenever $S = (S_k, \cdots, S_n)$ and $\bar{S} = (S_{k+1}, \cdots, S_n)$ s.t. $S_k \le S_{k+1} \le \cdots \le S_n$.

Indeed,

$$\mathbb{1}_{B^{(k)}_{S_k}} * \left(\delta_{x_k} \otimes \mathbb{1}_{R^{(k+1)}_{\bar{S}}}\right) = \int_{G_k} \mathbb{1}_{B^{(k)}_{S_k}}(x \cdot y^{-1}) \left(\delta_{y_k} \otimes \mathbb{1}_{R^{(k+1)}_{\bar{S}}}\right)(y)\, dy$$

$\forall x \in G_k$,

$$x = (x_k, x_{k+1}, \dots, x_n) = (x_k, 0, \cdots, 0) \cdot (0, x_{k+1}, \cdots, x_n) = (x_k) \cdot x'$$

with $x' \in G_{k+1}$

$$\rightsquigarrow \mathbb{1}_{B^{(k)}_{S_k}} * \left(\delta_{x_k} \otimes \mathbb{1}_{R^{(k+1)}_{\bar{S}}}\right)(x) = \int_{G_{k+1}} \mathbb{1}_{B^{(k)}_{S_k}}(x_k \cdot y') \, \mathbb{1}_{R^{(k+1)}_{\bar{S}}}(y'^{-1} \cdot x')\, dy'$$

Now if $x \in R^{(k)}_{cS_k}$ and $c > 0$ is small, then,

$$x = (x_k) \cdot x' \quad \text{with} \quad |x_k| \le (c S_k)^k \quad \text{and} \quad x' \in R^{(k+1)}_{c\bar{S}}$$

On supp $\mathbb{1}_{B^{(k)}_{S_k}}(x_k \cdot y')$, $\quad y' \in B^{(k+1)}_{S_{k+1}}$

{ If $y' \in B^{(k+1)}_{cS_{k+1}}$, $c > 0$ small, $\longrightarrow y'^{-1} \cdot x' \in R^{(k+1)}_{\bar{S}}$

$\Rightarrow \mathbb{1}_{B^{(k)}_{S_k}}(x_k \cdot y') = 1$ and $\mathbb{1}_{R^{(k+1)}_{\bar{S}}}(y'^{-1} \cdot x') = 1$ whenever $x \in R^{(k)}_{cS}$ and $y' \in B^{(k+1)}_{cS_{k+1}}$

$\Rightarrow$ If $x \in R^{(k)}_{cS}$, then, $\displaystyle\int_{G_{k+1}} \mathbb{1}_{B^{(k)}_{S_k}}(x_k \cdot y') \, \mathbb{1}_{R^{(k+1)}_{\bar{S}}}(y'^{-1} \cdot x')\, dy$

$$\ge \int_{y' \in B^{(k+1)}_{cS_{k+1}}} dy = m_{k+1}\left(B^{(k+1)}_{cS_{k+1}}\right)$$

$\Rightarrow$ (#1) holds.

Proceeding this way by induction gives

$$J_{R^{(1)}_{cS}} \leq C \; J_{B^{(1)}_{s_1}} * \left(\delta_{x_1} \otimes J_{B^{(2)}_{s_2}}\right) * \cdots * \left(\delta_{x_1,\dots,x_{n-1}} \otimes J_{B^{(n)}_{s_n}}\right)$$

whenever $s_1 \leq s_2 \leq \cdots \leq s_n$.

Thus,

$$\mathcal{M}(f)(x) = \sup_{R_{cS}} \left(|f| * J_{R_{cS}}\right)(x)$$

$$\lesssim \sup_{R_{cS}} |f| * J_{B^{(1)}_{s_1}} * \left(\delta_{x_1} \otimes J_{B^{(2)}_{s_2}}\right) * \cdots * \left(\delta_{x_1,\dots,x_{n-1}} \otimes J_{B^{(n)}_{s_n}}\right)(x)$$

$$\lesssim \tilde{\mathcal{M}}_n \circ \tilde{\mathcal{M}}_{n-1} \circ \cdots \circ \tilde{\mathcal{M}}_1 (f)$$

$$\Longrightarrow \|\mathcal{M} f\|_{L^p(G)} \lesssim \|\tilde{\mathcal{M}}_n \circ \tilde{\mathcal{M}}_{n-1} \circ \cdots \circ \tilde{\mathcal{M}}_1 (f)\|_{L^p(G)}$$

and

$$\left\| \left[\sum_j \left(\mathcal{M}(f_j)\right)^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)}$$

Thus, if $F_{t_1,\dots,t_n}(x)$ is a measurable function of $(t_1,\dots,t_n,x) \in (\mathbb{R}_+)^n \times \mathbb{R}^N$ we have

$$\left\| \left\{ \int_{(\mathbb{R}_+)^n} \left[\mathcal{M}\left(F_{t_1,\dots,t_n}(x)\right)\right]^2 dt_1 \cdots dt_n \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^N)}$$

$$\lesssim_{p,} \left\| \left\{ \int_{(\mathbb{R}_+)^n} \left|F_{t_1,\dots,t_n}(x)\right|^2 dt_1 \cdots dt_n \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^N)}$$

(1) Maximal op. $\mathcal{M}$ on G:

$$\mathcal{M}(f)(x) = \sup_{R_s = R_s^{(1)} \subseteq G_1 = G} \frac{1}{m(R_s)} \int_{R_s} |f(x \cdot y^{-1})| \, dy,$$

where
$$R_s^{(k)} = \left\{ x = (x_k, x_{k+1}, \cdots, x_n) \in G_k \,\middle|\, |x_k| < (s_k)^k, \ |x_{k+1}| < (s_{k+1})^{k+1}, \cdots, |x_n| < (s_n)^n \right\}$$

for $s = (s_k, s_{k+1}, \cdots, s_n)$, admissible (i.e. $s_k \le s_{k+1} \le \cdots \le s_n$).

Thm 3.1 (i) $\mathcal{M}$ is bounded on $L^p(G)$

(ii) $$\left\| \left[\sum_j (\mathcal{M}(f_j))^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \lesssim \left\| \left(\sum_j |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)}$$

(iii) $$\left\| \left\{ \int_{(\mathbb{R}_+)^n} \left[\mathcal{M}(F_{t_1,\cdots,t_n})(x) \right]^2 dt_1 \cdots dt_n \right\}^{1/2} \right\|_{L^p(G)} \lesssim \left\| \left(\int_{(\mathbb{R}_+)^n} |F_{t_1,\cdots,t_n}(\cdot)|^2 dt_1 \cdots dt_n \right)^{\frac{1}{2}} \right\|_{L^p(G)}$$

for any measurable function $F_{t_1,\cdots,t_n}(x)$ of $(t_1, \cdots, t_n, x)$ in $(\mathbb{R}_+)^n \times G$.

(2) Basic comparison function

Recall when $n=1$, then homogeneous dimension Q_1 of $\mathbb{R}^{a_1} = \mathbb{R}^N$ is $\boxed{Q_1 = N}$

& Poisson kernel
$$P_t(x) = t^{-N} \frac{1}{\left[1 + \left(\frac{|x|}{t}\right)^2\right]^{(N+1)/2}}$$
$$\approx t^{-N} \left(1 + \frac{|x|}{t}\right)^{-N-1}$$
$$= t \, (t + |x|)^{-N-1}$$
$$= t \, (t + |x|)^{-Q_1 - 1}$$

When $n \geq 2$, we hope we could have some function $\Gamma_t(x)$ s.t.

① $$\sup_{t \in (\mathbb{R}_+)^n} |f * \Gamma_t(x)| \le c \, \mathcal{M}(f)(x)$$

② $|\mathcal{K} * \Phi_t(x)| \lesssim \Gamma_t(x)$ for some appropriate function $\Phi_t(x)$ which will be used to define square functions.

Recall Flag kernel $\mathcal{K}$ s.t.
$$|\partial_x^\alpha \mathcal{K}(x)| \lesssim \prod_{j=1}^{n} \left(N_1(x_1) + \cdots + N_j(x_j) \right)^{-Q_j - [\![\alpha_j]\!]}$$

when $x_1 \neq 0$

So we choose the basic comparison function $\Gamma_t(x)$ as

$$\Gamma_t(x) = t_1 t_2 \cdots t_n \prod_{j=1}^{n} \left(t_1 + \cdots + t_j + N_1(x_1) + \cdots + N_j(x_j) \right)^{-Q_j - 1}$$

Here, $N_j(x_j) \approx |x_j|^{1/j}$ and $Q_j = \sum_{\ell \in J_j} d_\ell = j \# J_j = j\, a_j$.

Thm 3.2 $\sup_{t \in (\mathbb{R}_+)^n} |f * \Gamma_t(x)| \leq C\, \mathcal{M}(f)(x)$

Pf It suffices to consider $t = (t_1, \cdots, t_n)$ that $t_1 \leq t_2 \leq \cdots \leq t_n$

if not, set $\tilde{t}_j := t_1 + \cdots + t_j$, $1 \leq j \leq n$. Then, $\tilde{t}_j \leq \tilde{t}_{j+1}$

but, $k(t_1 + \cdots + t_k) \geq \tilde{t}_1 + \cdots + \tilde{t}_k$,

$$\rightsquigarrow t_1 + \cdots + t_k + N_1(x) + \cdots + N_k(x) \geq \frac{\tilde{t}_1 + \cdots + \tilde{t}_k}{k} + N_1(x) + \cdots + N_k(x)$$

$$\geq \frac{1}{k}\left(\tilde{t}_1 + \cdots + \tilde{t}_k + N_1(x) + \cdots + N_k(x) \right)$$

$\rightsquigarrow \Gamma_t(x) \leq \Gamma_{\tilde{t}}(x)$ for $t = (t_1, \cdots, t_n)$ and $\tilde{t} = (\tilde{t}_1, \cdots, \tilde{t}_n)$

Now we decompose the space $G = \mathbb{R}^N$ into a preliminary dyadic partition as follows: fix $t = (t_1, \cdots, t_n)$, for each $J = (j_1, \cdots, j_n) \in \mathbb{N}^n$ set

$$R_J := \left\{ x \in \mathbb{R}^N \;\middle|\; 2^{j_k - 1} < \frac{N_1(x) + \cdots + N_k(x)}{t_1 + t_2 + \cdots + t_k} \leq 2^{j_k} \text{ for } k = 1, 2, \cdots, n \right\}$$

with the understanding that if $j_k = 0$, the ineq. should be taken to be

$$\frac{N_1(x) + \cdots + N_k(x)}{t_1 + \cdots + t_k} \leq 1$$

Then,

$$G = \bigcup_{J = (j_1, \cdots, j_n) \in \mathbb{N}^n} R_J .$$

However, in general, each R_J is not comparable to an acceptable rectangle

Now, if R_J is non-empty, then, since $t_1 \leq t_2 \leq \cdots \leq t_n$ we have

$$t_k 2^{j_k} \approx (t_1 + \cdots + t_k) 2^{j_k} \approx N_1(x) + \cdots + N_k(x), \quad 1 \leq k \leq n$$

we set that for sufficiently large C,

$$S_k^J \triangleq C^k t_k 2^{j_k}$$

then $S_1^J \leq S_2^J \leq \cdots \leq S_n^J$

Now define

$$R_J^* = \{x \in \mathbb{R}^N \mid N_k(x) \le S_k^J, \ k=1,\cdots,n\} \quad \text{for those } J \ne \emptyset$$

Then, clearly

$$R_J \subset R_J^*$$

$\forall x \in R_J$, $N_k(x) \le N_1(x) + \cdots + N_k(x) \le (t_1 + \cdots + t_k) 2^{j_k} \approx t_k 2^{j_k}$

$$\le C^k t_k 2^{j_k} = S_k^J$$

$\rightsquigarrow x \in R_J^*$

Thus,

$$|f * \Gamma_t(x)| = \left| \int_G f(x \cdot y^{-1}) \Gamma_t(y) dy \right| \qquad \because G = \bigcup_J R_J$$

$$= \left| \sum_{J \in \mathbb{N}^n} \int_{R_J} f(x \cdot y^{-1}) \Gamma_t(y) dy \right|$$

$$\le \sum_{J \in \mathbb{N}^n} \int_{R_J^*} |f(x \cdot y^{-1})| \Gamma_t(y) dy$$

Since $t_k \approx 2^{-j_k} S_k^J$ and $N_1 + \cdots + N_k \approx 2^{j_k}(t_1 + \cdots + t_k)$

$$\Gamma_t(y) = t_1 \cdots t_n \prod_{k=1}^n \left(t_1 + \cdots + t_k + N_1(y) + \cdots + N_k(y)\right)^{-Q_k - 1}$$

$$= t_1 \cdots t_n \prod_{k=1}^n (t_1 + \cdots + t_k)^{-Q_k - 1} \left(1 + \frac{N_1(y) + \cdots + N_k(y)}{t_1 + \cdots + t_k}\right)^{-Q_k - 1}$$

$$= t_1^{-Q_1} \frac{t_2}{(t_1 + t_2)^{Q_2+1}} \cdots \frac{t_n}{(t_1 + \cdots + t_n)^{Q_n+1}} \prod_{k=1}^n \left(1 + \frac{N_1(y) + \cdots + N_k(y)}{t_1 + \cdots + t_k}\right)^{-Q_k - 1}$$

$$\le t_1^{-Q_1} t_2^{-Q_2} \cdots t_n^{-Q_n} \prod_{k=1}^n \left(1 + \frac{N_1(y) + \cdots + N_k(y)}{t_1 + \cdots + t_k}\right)^{-Q_k - 1}$$

$\downarrow \because t_k \approx 2^{-j_k} S_k^J$

$$\approx \left(\prod_{k=1}^n 2^{-j_k} S_k^J\right)^{-Q_k} \prod_{k=1}^n \left(1 + 2^{j_k}\right)^{-Q_k - 1}$$

$$\approx \prod_{k=1}^n (S_k^J)^{-Q_k} \prod_{k=1}^n 2^{j_k Q_k + j_k(-Q_k - 1)}\} = \prod_{k=1}^n (S_k^J)^{-Q_k} \prod_{k=1}^n 2^{-j_k Q_k}$$

$$\prod_{k=1}^n (S_k^J)^{-Q_k} = c\, m(R_J^*)$$

Thus,

$$|f * \Gamma_t(y)| \lesssim \sum_{J \in \mathbb{N}^n} \prod_{k=1}^n 2^{-j_k Q_k} \frac{1}{m(R_J^*)} \int_{R_J^*} |f(x \cdot y^{-1})| dy$$

$$\lesssim \mathcal{M}(f)(x) \quad \square$$

(3) Kernel estimates & operator estimates

Suppose $\varphi^{(k)} \in C_c^\infty(G_k)$ is supported on the unit ball of G_k with

$$\int_{G_k} \varphi^{(k)}(x)\, dx = 0$$

For any $\tau > 0$ and $x \in G_k$ we set

$$\varphi_\tau^{(k)}(x) = \tau^{-(Q_k + Q_{k+1} + \cdots + Q_n)} \varphi(\tau^{-1} \cdot x) \quad \text{with } Q_k = k a_k$$

and let $\tilde{\varphi}_\tau^{(k)}$ be the corresponding distributions lifted to the full group; i.e.

$$\tilde{\varphi}_\tau^{(k)}(x) := \delta_{x_1, \cdots, x_{k-1}} \otimes \varphi_\tau^{(k)} \quad \text{for any } x \in G$$

and set

$$\Phi_{\boldsymbol{t}} := \tilde{\varphi}_{t_1}^{(1)} * \tilde{\varphi}_{t_2}^{(2)} * \cdots * \tilde{\varphi}_{t_n}^{(n)} \quad \text{for } \boldsymbol{t} = (t_1, \cdots, t_n)$$

&

$$\Phi_{\boldsymbol{t}}^* := \tilde{\varphi}_{t_n}^{(n)} * \tilde{\varphi}_{t_{n-1}}^{(n-1)} * \cdots * \tilde{\varphi}_{t_1}^{(1)} \quad \text{for } \boldsymbol{t} = (t_1, \cdots, t_n).$$

<u>Thm 3.3</u> Suppose $\mathcal{K}$ is a flag kernel. Then, for $\boldsymbol{t} = (t_1, \cdots, t_n) \in (\mathbb{R}_+)^n$ and $x \in G$

(1) $|\mathcal{K} * \Phi_{\boldsymbol{t}}(x)| \lesssim \Gamma_{\boldsymbol{t}}(x)$, $\quad |\Phi_{\boldsymbol{t}}^* * \mathcal{K}(x)| \lesssim \Gamma_{\boldsymbol{t}}(x)$

for $\boldsymbol{t} = (t_1, \cdots, t_n) \in (\mathbb{R}_+)^n$ and $x \in G$

(2) $(f * \mathcal{K}) * \Phi_{\boldsymbol{t}}(x)| \lesssim \mathcal{M}(f)(x)$,

$|(f * \Phi_{\boldsymbol{t}}^*) * \mathcal{K}(x)| \lesssim \mathcal{M}(f)(x)$

(3) $|(f * \Phi_{\boldsymbol{s}}^* * \mathcal{K} * \Phi_{\boldsymbol{t}})(x)| \lesssim \gamma(\boldsymbol{s}, \boldsymbol{t})\, \mathcal{M}(f)(x)$,

where $\gamma(\boldsymbol{s}, \boldsymbol{t}) \lesssim \prod_{k=1}^{n} \left(\min\left(\frac{s_k}{t_k}, \frac{t_k}{s_k}\right) \right)^\delta$ for some $\delta > 0$

$\boxed{\delta = 1/n^2}$

(4). Square function.

Each G_k is a homogeneous group with family of dilations $\delta_\lambda : x \mapsto \lambda \cdot x$ for $x \in G_k$ and so there exists a finite-dim. inner product space V_k and a pair of V_k-valued functions, $\varphi^{(k)}$ and $\psi^{(k)}$, with $\varphi^{(k)} \in C_c^\infty(G_k)$ supported in the unit ball, $\psi^{(k)} \in \mathcal{S}(G_k)$ such that

$$\int_{G_k} \varphi^{(k)}(x)\,dx = \int_{G_k} \psi^{(k)}(x)\,dx = 0$$

and

$$\int_0^\infty \psi_\tau^{(k)}(x \cdot y^{-1}) \cdot \varphi_\tau^{(k)}(y)\,\frac{d\tau}{\tau} = \delta_0 .$$

Here, $\varphi_\tau^{(k)}(y) = \tau^{-(Q_k + \cdots + Q_n)} \varphi^{(k)}(\tau^{-1} \cdot x)$ for $\tau > 0$ and $x \in G_k$.

We write

$$\tilde{\varphi}_\tau^{(k)}(x) = \delta_{x_1, \dots, x_{k-1}} \otimes \varphi_\tau^{(k)} \quad \text{for } x \in G \text{ and } \tau > 0$$

$$\tilde{\psi}_\tau^{(k)}(x) = \delta_{x_1, \dots, x_{k-1}} \otimes \psi_\tau^{(k)} \quad \text{for } x \in G \text{ and } \tau > 0$$

$$\Phi_t(x) = \left(\tilde{\varphi}_{t_1}^{(1)} * \tilde{\varphi}_{t_2}^{(2)} * \cdots * \tilde{\varphi}_{t_n}^{(n)}\right)(x) \quad \text{for } x \in G \text{ and } t_1, \dots, t_n > 0$$

$$\Phi_t^*(x) = \left(\tilde{\varphi}_{t_n}^{(n)} * \tilde{\varphi}_{t_{n-1}}^{(n-1)} * \cdots * \tilde{\varphi}_{t_1}^{(1)}\right)(x) \quad \text{for } x \in G \text{ and } t_1, \dots, t_n > 0$$

$$\Psi_t(x) = \left(\tilde{\psi}_{t_1}^{(1)} * \cdots * \tilde{\psi}_{t_n}^{(n)}\right)(x) \quad \text{for } x \in G \text{ and } t_1, \dots, t_n > 0,$$

$$\Psi_t^*(x) = \left(\tilde{\psi}_{t_n}^{(n)} * \cdots * \tilde{\psi}_{t_1}^{(1)}\right)(x) \quad \text{for } x \in G \text{ and } t_1, \dots, t_n > 0.$$

Here, $\Phi_t(x)$, $\Phi_t^*(x)$, $\Psi_t(x)$, $\Psi_t^*(x)$ are V-valued functions, $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

Finally we set

$$S_\Phi(f)(x) := \left(\int_{(\mathbb{R}_+)^n} |f * \Phi_t(x)|^2 \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}\right)^{\frac{1}{2}},$$

$$S_\Phi^\#(f)(x) = \left(\int_{(\mathbb{R}_+)^n} |\mathcal{M}(f * \Phi_t^*)(x)|^2 \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}\right)^{1/2}$$

& $S_\psi(f)(x) = \left(\int_{(\mathbb{R}_+)^n} |f * \Psi_t(x)|^2 \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}\right)^{\frac{1}{2}}$, $\quad S_\psi^\#(f)(x) = \left(\int_{(\mathbb{R}_+)^n} |\mathcal{M}(f * \Psi_t^*)(x)|^2 \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}\right)^{\frac{1}{2}}$.

Thm 3.4 $\|f\|_{L^p(G)} \approx \|S_\Phi(f)\|_{L^p} \approx \|S_\Phi(f)\|_{L^p(G)}$

$f(x)=\int_{(\mathbb{R}_+)^n} f * \Phi_t * \Phi_t^* \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n} \quad \forall x \in G$

Thm 3.1 – Thm 3.4

$\Rightarrow$ op. $T: f \mapsto f * \mathcal{K}$ with flag kernel $\mathcal{K}$ is bounded on $L^p(G)$ for $1<p<\infty$.

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4. Composition of two op. $T_{\mathcal{K}_j}: f \mapsto f * \mathcal{K}_j$ with flag kernels $\mathcal{K}_j$, $j=1,2$.

Q If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two flag kernels on G, then the composition

$$T_{\mathcal{K}_2} \circ T_{\mathcal{K}_1} \overset{?}{=\!=} T_{\mathcal{K}_3} \quad \text{with flag kernel } \mathcal{K}_3 \text{ on } G$$

Formally,

$$(T_{\mathcal{K}_2} \circ T_{\mathcal{K}_1})(f) \overset{?}{=} T_{\mathcal{K}_2}\left(T_{\mathcal{K}_1}(f)\right) = \left(T_{\mathcal{K}_1}(f)\right) * \mathcal{K}_2 \overset{?}{=} f * \mathcal{K}_1 * \mathcal{K}_2$$

so, $T_{\mathcal{K}_2} \circ T_{\mathcal{K}_1}$ should be given by convolution with $\mathcal{K}_1 * \mathcal{K}_2$.

Problem For $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{S}'(\mathbb{R}^N)$, $\mathcal{K}_1 * \mathcal{K}_2$ does not make sense unless one of them have compact support

$\forall \varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\langle \mathcal{K}_1 * \mathcal{K}_2, \varphi \rangle = \langle \mathcal{K}_1(\xi) \otimes \mathcal{K}_2(\eta), \varphi(\xi+\eta) \rangle,$$

We have already known that

$T_{\mathcal{K}}(f) \in L^p(\mathbb{R}^N)$ for any $f \in \mathcal{S}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$

and the mapping

$$T_{\mathcal{K}}: \mathcal{S}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$$

has a (unique) continuous extension to a mapping of $L^p(\mathbb{R}^N)$ to itself

$\rightsquigarrow$ we can define $T_{\mathcal{K}_2} \circ T_{\mathcal{K}_1}$ as the composition of two mappings from $L^P(\mathbb{R}^N)$ to itself.

Lem Suppose that $\mathcal{K}$ is a flag kernel and that $\mathcal{K} = \sum_{I \in \mathcal{E}_n} [\varphi^I]_I$. Then, for all $f \in L^P(\mathbb{R}^N)$, $1 < p < \infty$,

$$\lim_{F \nearrow \mathcal{E}_n} \left\| \sum_{I \in F} \left(T_{[\varphi^I]_I}(f) - T_{\mathcal{K}}(f) \right) \right\|_{L^P} = 0$$

Pf Since $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^P(\mathbb{R}^N)$ and since $T_{\mathcal{K}}$ is bounded on $L^P(\mathbb{R}^N)$ it suffices to show this lemma for $f \in \mathcal{S}(\mathbb{R}^N)$.

For $f \in \mathcal{S}(\mathbb{R}^N)$ and $\varphi^I \in \mathcal{S}(\mathbb{R}^N)$ we have known that

$$\langle \mathcal{K}, f \rangle = \lim_{F \nearrow \mathcal{E}_n} \langle \sum_{I \in F} [\varphi^I]_I, f \rangle$$

$\square$

Thm (Nagel-Ricci-Stein-Wainger, 2012)
Let $\mathcal{F}_1, \mathcal{F}_2$ be two standard flags on $\mathbb{R}^N$.

$\mathcal{F}_1$: $(0) \subset \mathbb{R}^{a_n} \subset \mathbb{R}^{a_{n-1}} \oplus \mathbb{R}^{a_n} \subset \dots \subset \mathbb{R}^{a_2} \oplus \dots \oplus \mathbb{R}^{a_n} \subset \mathbb{R}^N$

$\mathcal{F}_2$: $(0) \subset \mathbb{R}^{b_n} \subset \mathbb{R}^{b_{n-1}} \oplus \mathbb{R}^{b_n} \subset \dots \subset \mathbb{R}^{b_2} \oplus \dots \oplus \mathbb{R}^{b_n} \subset \mathbb{R}^N$.

For each $j=1,2$. let $\mathcal{K}_j$ be a flag kernel adapted to the flag $\mathcal{F}_j$.
Then, $T_{\mathcal{K}_2} \circ T_{\mathcal{K}_1}$ is an op. with flag kernel adapted to a flag $\mathcal{F}_0$ which is the coarest flag on $\mathbb{R}^N$

Pf $\mathcal{K}_1 = \sum_{I \in \mathcal{E}_n} [\varphi^I]_I$, $\quad \mathcal{K}_2 = \sum_{J \in \mathcal{E}_m} [\psi^J]_J$,
where $\{\varphi^I\}_{I \in \mathcal{E}_n}$, $\{\psi^J\}_{J \in \mathcal{E}_m}$ are two families of functions with uniformly bounded semi-norms and with strong cancellation relative to flags $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

If $f, g \in \mathcal{S}(\mathbb{R}^N)$, then,

$$T_{\pi_1}(f) = \lim_{F_1 \nearrow \mathcal{E}_n} \sum_{I \in F_1} f * [\varphi^I]_I$$

&

$$T_{\pi_2}(f) = \lim_{F_2 \nearrow \mathcal{E}_m} \sum_{J \in F_2} f * [\psi^J]_J$$

where the limits are in $L^p(\mathbb{R}^N)$ and are taken over finit sets $F_1 \subset \mathcal{E}_n$, $F_2 \subset \mathcal{E}_m$.

For any $F_1 \subset \mathcal{E}_n$ finite subset, $\sum_{I \in F_1} f * [\varphi^I]_I \in \mathcal{S}(\mathbb{R}^N)$ for $f, \varphi^I \in \mathcal{S}$.

$$\Rightarrow T_{\pi_2}(T_{\pi_1}(f)) = T_{\pi_2}\left(\lim_{F_1 \nearrow \mathcal{E}_n} f * [\varphi^I]_I\right)$$

& $\because$ T_{π_2} is bdd in $L^p(\mathbb{R}^N)$

$$= \lim_{F_1 \nearrow \mathcal{E}_n} T_{\pi_2}(f * [\varphi^I]_I)$$

$$= \lim_{F_1 \nearrow \mathcal{E}_n} \lim_{F_2 \nearrow \mathcal{E}_m} f * [\varphi^I]_I * [\psi^J]_J$$

One gets that there exists $\Theta^K \in C_c^\infty(\mathbb{R}^N)$, normalized relative to $\{\varphi^I\}$ & $\{\psi^J\}$ so that

$$\sum_{(I,J) \in \mathcal{E}_n} [\varphi^I]_I * [\psi^J]_J = [\Theta^K]_K$$

and Θ^K has weak cancellation relative to the decomposition of $\mathbb{R}^N$

$\rightsquigarrow \sum_K [\Theta^K]_K$ is a flag kernel relative to this decomposition of $\mathbb{R}^N$. $\square$