Global pseudodifferential calculus on manifolds.

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Outline

1. Classical ΨDOs.
2. Safarov Calculus.
3. $L^p - L^p$ bounds.
4. Relation with other global pseudodifferential calculus and other perspective of research.

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Motivation for ΨDOs

Consider a partial linear differential operator with constant coefficients

$$P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha,$$

we are interested in solving the equation

$$Pu = f.$$

Using Fourier transform we find that the solution is

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{p(\xi)} \hat{f}(\xi)\, d\xi.$$

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We can think on operators as defined by functions, called symbols, through integral expressions:

$$p(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi.$$

- Class $S^m_{1,0}$ (Kohn-Nirenberg [3]): $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, for any multi-indices α, β, there exist a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|}.$$

- Class $S^m_{\rho,\delta}$ (Hörmander [2]): $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, let $0 \leq \delta < \rho \leq 1$, for any multi-indices α, β, there exist a constant $C_{\alpha,\beta}$ such that

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Properties of $S^m_{\rho,\delta}$

Proposition

- If $p \in S^{m_1}_{\rho,\delta}$ and $q \in S^{m_2}_{\rho,\delta}$, set $m = \max(m_1, m_2)$, then

$$pq \in S^{m_1+m_2}_{\rho,\delta} \text{ and } a + b \in S^m_{\rho,\delta}.$$

- Moreover, $\partial^\alpha_\xi p \in S^{m-\rho|\alpha|}_{\rho,\delta}$, $\partial^\beta_x p \in S^{m+\delta|\beta|}_{\rho,\delta}$ for all $p \in S^m_{\rho,\delta}$.

$\Psi^m_{\rho,\delta} := OPS^m_{\rho,\delta}$

Definition

If $p(x,\xi) \in S^m_{\rho,\delta}$, then the pseudodifferential operator

$$p(x,D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi$$

belongs to $\Psi^m_{\rho,\delta} := OPS^m_{\rho,\delta}$.

We usually would like to have:

- Continuity in "nice" spaces.
- Adjoint to be part of the operator classes (amplitudes).
- Products to be part of the operator classes (amplitudes).

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Continuity

Theorem

Let $0 \leq \delta < \rho \leq 1$. If $p \in S^m_{\rho,\delta}$, then

$$p(x,D) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is continuous. The same holds true for the Schwartz space $S(\mathbb{R}^n)$.

Calculus: Adjoint

Theorem

Let $0 \leq \delta < \rho \leq 1$. If $p(x, D) \in \Psi^m_{\rho,\delta}$, then

$$p(x, D)^* \in \Psi^m_{\rho,\delta}$$

and $p(x, D)^* = p^*(x, D)$ with

$$p^*(x, \xi) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{p(x, \xi)}.$$

Calculus: Products

Theorem

Let $0 \leq \delta < \rho \leq 1$. If $p(x, D) \in \Psi^{m_1}_{\rho,\delta}$ and $q(x, D) \in \Psi^{m_2}_{\rho,\delta}$, then

$$p(x, D)q(x, D) \in \Psi^{m_1+m_2}_{\rho,\delta}$$

and $p(x, D)q(x, D) = r(x, D)$ with

$$r(x, \xi) \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x, \xi) \partial_x^\alpha q(x, \xi).$$

Then one can study many different properties of these operators: Inequalities, functional calculus, spectral properties, etc.

Calculus: Products

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ΨDOs on manifolds

One can not use the same definition on a manifold M since Fourier transform is not globally well-defined. Then one can use local structure, i.e., if $P : C^\infty(M) \to C^\infty(M)$, (U, κ) a chart, then one would like

$$\kappa_*(\phi P \psi) = (\kappa^*)^{-1} \phi P \psi \kappa^* \in \Psi^m_{\rho,\delta}(\kappa(U))$$

for all $\phi, \psi \in C_c^\infty(U)$.

Remark

In this case the symbol p will be a function defined on T^*M.

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Change of coordinates

Theorem

Let $0 \leq \delta < \rho \leq 1$ and $1 - \rho \leq \delta$. Let U_1 and U_2 be open sets in $\mathbb{R}^n$ and let $\phi : U_1 \to U_2$, $\Phi : U_1 \to GL(n)$ be smooth maps. Then

$$p_1(x, \xi) = p_2(\phi(x), \Phi(x)\xi)$$

is in $S^m_{\rho,\delta}(U_1)$ for every $p_2 \in S^m_{\rho,\delta}(U_2)$.

Remark

$0 \leq \delta < \rho \leq 1$ and $1 - \rho \leq \delta$ implies that $\rho > 1/2$.

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Problem on having global calculus

One can try to avoid using Fourier transform by

$$p(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x, \xi) u(y)\, dy\, d\xi,$$

but this still not globally defined because the phase function

$$\varphi(x, y, \xi) = (x - y) \cdot \xi$$

is not invariant.

Some geometry: Connection

A linear connection ∇ on $T^*M \to M$ is a splitting of the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & VT^*M \arrow[r] & TT^*M \arrow[r] & HT^*M \arrow[r] \arrow[l, bend right, dashed, "\nabla"'] & 0. \end{tikzcd}$$

Or a covariant derivative is a $\mathbb{R}-$linear map $\nabla : \Gamma(TM) \times \Gamma(T^*M) \to \Gamma(T^*M)$ such that

1. $\nabla_v(fs) = df\, s + f\nabla_v(s)$ for all smooth function f.
2. $\nabla_v(a_1 s_1 + a_2 s_2) = a_1 \nabla_v s_1 + a_2 \nabla_v s_2$.

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Some geometry: Connection

This distribution HT^*M is generated by the horizontal lifts of the standard vector fields $v = \sum v^k(y)\partial_{y^k} \in \mathfrak{X}(M)$, which are defined as follows

$$\nabla_v = \sum_k v^k(y)\partial_{y^k} + \sum_{i,j,k} \Gamma^i_{kj}(y)v^k(y)\eta_i\partial_{\eta_j}.$$

We can extend this derivatives to any tensor.

On the other hand, the distribution VT^*M is generated by the vector fields $\partial_{\eta_1}, \ldots, \partial_{\eta_1}$.

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Some geometry: Geodesics

- A geodesic is a curve $\gamma(t)$ such that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.
- Given a neighborhood U_x of x, we denote by $\gamma_{y,x}(t)$ the shortest geodesic joining x and $y \in U_x$. We will use de notation $z_t = \gamma_{y,x}(t)$.
- If we are in a normal coordinate system y^k we have that

$$\gamma^k_{y,x}(t) = x^k + t(y^k - x^k), \quad \dot{\gamma}^k_{y,x}(t) = (y^k - x^k).$$

- Let $\Phi_{y,x} : T^*_x M \to T^*_y M$ the parallel transport along $\gamma_{y,x}(t)$ and $\Upsilon_{y,x} = \left|\det \Phi_{y,x}\right|$.

Some geometry: Densities

The κ-density bundle is defined as the associated bundle of the following representation of $\mathrm{GL}(n)$:

$$\begin{aligned} \mathrm{GL}(n) &\xrightarrow{\rho} \mathrm{GL}(1) \\ A &\mapsto |\det A|^{-\kappa}, \end{aligned}$$

i.e. the bundle Ω^{κ} is defined as

$$\begin{array}{c} \Omega^{\kappa} := \mathrm{Fr}(M) \times_{\rho} V \\ \big\downarrow \\ M \end{array}$$

$$\Omega^{\kappa} \times \Omega^{1-\kappa} \xrightarrow{\langle\cdot,\cdot\rangle = \int} \mathbb{R}.$$

Safarov approach: Symbols

Definition

The space $S^m_{\rho,\delta}(\nabla)$ denotes the class of functions $a \in C^\infty(T^*M)$ such that the estimates

$$\left|\partial^\alpha_\eta \nabla_{i_1} \dots \nabla_{i_q} a(y,\eta)\right| \leq C_{K,\alpha,i_1,\dots,i_q} \langle\eta\rangle_y^{m+\delta q-\rho|\alpha|}$$

holds in any coordinates y, for all α and $i_1,\dots,i_q$. Here y runs over a compact set $K \subset M$, $\langle\eta\rangle_y := (1 + w^2(y,\eta))^{1/2}$ where $w \in C^\infty(T^*M \setminus \{0\})$ is homogeneous in η of degree 1.

Here we also assume $0 \leq \delta < \rho \leq 1$.

Safarov [5] approach: Symbols

Same properties:

Proposition

- If $a \in S^{m_1}_{\rho,\delta}(\nabla)$ and $b \in S^{m_2}_{\rho,\delta}(\nabla)$, set $m = \max(m_1, m_2)$, then

$$ab \in S^{m_1+m_2}_{\rho,\delta}(\nabla) \text{ and } a + b \in S^{m}_{\rho,\delta}(\nabla).$$

- Moreover, $\partial^\alpha_\eta a \in S^{m-\rho|\alpha|}_{\rho,\delta}(\nabla)$, $\nabla_{v_1} \cdots \nabla_{v_q} a \in S^{m+\delta q}_{\rho,\delta}(\nabla)$ for all $a \in S^m_{\rho,\delta}(\nabla)$ and any $v_1, \ldots, v_q \in \mathfrak{X}(M)$.

Safarov [5] approach: Phase functions

Definition

Let V be a sufficiently small neighborhood of Δ in $M \times M$. We introduce the phase functions

$$\varphi_\tau(x,\zeta,y) = -\langle \dot{\gamma}_{y,x}(\tau), \zeta \rangle, \text{ where } (x,y) \in V, \tau \in [0,1], \zeta \in T^*_{z_\tau} M.$$

- In n.c.s for all $\tau \in [0,1]$.

$$\varphi_\tau(x,\zeta,y) = (x-y)\cdot\zeta.$$

- For all $\tau, s \in [0,1]$

$$\varphi_\tau(x,\zeta,y) = \varphi_{1-\tau}(y,\zeta,x), \quad \varphi_\tau(x,\zeta,y) = \varphi_s(x, \Phi_{z_s,z_\tau}\zeta, y).$$

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Safarov [5] approach: ΨDOs

Definition

Let $A : \Gamma_c(\Omega^\kappa) \to \Gamma(\Omega^\kappa)$ be a linear operator with Schwartz kernel $\mathscr{A}(x,y)$, i.e, $\langle Au, v\rangle = \langle \mathscr{A}, uv\rangle$. We say that A is pseudodifferential if

1. $\mathscr{A}(x,y)$ is smooth in $(M \times M) \setminus \Delta$.
2. On a neighborhood V of Δ the Schwartz kernel is represented by an oscillatory integral of the form

$$\mathscr{A}(x,y) = \frac{1}{(2\pi)^n} p_{\kappa,\tau} \int e^{i\varphi_\tau(x,\zeta,y)} a(z_\tau,\zeta)\, d\zeta, \text{ for } (x,y) \in V$$

where $p_{\kappa,\tau} = p_{\kappa,\tau}(x,y) = \Upsilon^{1-\kappa}_{y,z_\tau} \Upsilon^{-\kappa}_{z_\tau,x}$.
We denote by $\Psi^m_{\rho,\delta}(\Omega^\kappa, \nabla, \tau)$ this class of ΨDOs.

Safarov [5] approach: Examples

1. Let (M, g) be a pseudo-Riemannian manifold, ∇_{LC} its Levi-Civita connection, then

$$\sigma_{\Delta_g}(x, \xi) = -|\xi|_x^2 = \frac{1}{3}S(x).$$

2. Suppose M is parallelizable by $v_1, \dots, v_n$. Consider the operator

$$A^{\alpha}_{(\kappa)}(y, D_y) = \sum_{i_1, \dots, i_q} A^{(\kappa)}_{i_1} \cdots A^{(\kappa)}_{i_1},$$

then ${}_s\sigma_{A^{\alpha}_{(\kappa)}}(x, \xi) = i^{|\alpha|}\sigma^{\alpha}(x, \xi)$ for all $s, \kappa \in \mathbb{R}$, where

$$\sigma^{\alpha} = (\sigma_1)^{\alpha_1} \dots (\sigma_n)^{\alpha_n}, \quad \sigma_l = \sigma_l(x, \xi) = \langle v_l(x), \xi \rangle .$$

For anisotropic version see Shargorodsky [6].

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For anisotropic version see Shargorodsky [6].

Safarov [5] approach: Nice properties

- For all α $\partial_\zeta^\alpha e^{i\varphi_\tau(x,\zeta,y)} = (-1)^{|\alpha|}\dot\gamma_{x,y}^\alpha e^{i\varphi_\tau(x,\zeta,y)}$.
- Let $a \in S^m_{\rho,\delta}(\nabla)$, then for all non-negative integers q

$$a(y,\Phi_{y,x}\xi) = \sum_{|\alpha|\le q} \frac{1}{\alpha!}\dot\gamma_{x,y}^\alpha \nabla_x^\alpha a(x,\xi) + \sum_{|\alpha|=q+1} \dot\gamma_{x,y}^\alpha \nabla_x^\alpha \tilde a_\alpha(y,x,\xi)$$

where $\tilde a_\alpha \in S^{m-\delta|\alpha|}_{\rho,\delta'}(\nabla)$ and $\delta' = \max\{\delta, 1-\rho\}$.

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- Let $a \in S_{\rho,\delta}^{m}(\nabla)$, then for all non-negative integers q

$$a(y, \Phi_{y,x}\xi) = \sum_{|\alpha| \leq q} \frac{1}{\alpha!} \dot{\gamma}_{x,y}^{\alpha} \nabla_{x}^{\alpha} a(x, \xi) + \sum_{|\alpha| = q+1} \dot{\gamma}_{x,y}^{\alpha} \nabla_{x}^{\alpha} \tilde{a}_{\alpha}(y, x, \xi)$$

where $\tilde{a}_{\alpha} \in S_{\rho,\delta'}^{m-\delta|\alpha|}(\nabla)$ and $\delta' = \max\{\delta, 1-\rho\}$.

Safarov [5] approach: Adjoint

Theorem

If $A \in \Psi^m_{\rho,\delta}\left(\Omega^\kappa, \nabla\right)$ then $A^* \in \Psi^m_{\rho,\delta}\left(\Omega^{1-\kappa}, \nabla\right)$ and

$$\sigma_{A^*,\tau}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(1-2\tau)^{|\alpha|} D_\xi^\alpha \nabla_x^\alpha \overline{\sigma_{A,\tau}(x,\xi)},$$

as $\langle \xi \rangle_x \to \infty$.

Safarov [5] approach: Products

Theorem

Let $A \in \Psi^{m_1}_{\rho,\delta}(\Omega^\kappa, \nabla)$, $B \in \Psi^{m_2}_{\rho,\delta}(\Omega^\kappa, \nabla)$, and let at least one of these ψDOs be properly supported. Assume that at least one of the following conditions is fulfilled:

1. $\rho > \frac{1}{2}$;
2. the connection ∇ is symmetric and $\rho > \frac{1}{3}$;
3. the connection ∇ is flat.

Then $AB \in \Psi^{m_1+m_2}_{\rho,\delta}(\Omega^\kappa, \nabla)$ and

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha,\beta,\gamma} \frac{1}{\alpha!}\frac{1}{\beta!}\frac{1}{\gamma!} P^{(\kappa)}_{\beta,\gamma}(x,\xi) D^{\alpha+\beta}_\xi \sigma_A(x,\xi) D^\gamma_\xi \nabla^\alpha_x \sigma_B(x,\xi),$$

as $\langle \xi \rangle_x \to \infty$.

Safarov [5] approach: Other results

- L^2-estimates.
- Parametrices.
- Functional calculus for powers of the laplacian.

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Fefferman[1] L^p−estimates

Theorem

a) Let $\sigma(x,\xi) \in S^{-\beta}_{1-a,\delta}(R^n)$ with $0 \leqq \delta < 1-a < 1$ and $\beta < na/2$. Then $\sigma(x,D)$ is bounded on L^p for

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leqq \gamma = \frac{\beta}{n}\left[\frac{n/2+\lambda}{\beta+\lambda}\right], \lambda = \frac{na/2-\beta}{1-a}.$$

b) If $|1/p - 1/2| > \gamma$, then the symbol

$$\sigma(x,\xi) = \sigma_{\alpha\beta}(\xi) = \frac{e^{i|\xi|^\alpha}}{1+|\xi|^\beta} \in S^{-\beta}_{1-a.0}$$

provides an operator $\sigma_{\alpha\beta}(D)$ unbounded on L^p.

Fefferman[1] L^p-estimates

c) Let $\sigma(x,\xi) \in S^{-na/2}_{1-a\delta}$, so that the critical L^p space is L^1. Although $\sigma(x,D)$ is unbounded on L^1, it is bounded on the Hardy space H^1.

L^p-estimates for Safarov ΨDOs

We can define naturally the intrinsic L^p spaces on manifolds as follows:

$$L^p\left(M, \Omega^{1/p}\right) := \left\{\lambda \in \Omega^{1/p} : \left(\int_M |\lambda|^p\right)^{1/p} < \infty\right\},$$

$$L^\infty\left(M, \Omega^0\right) := \left\{f \in \Omega^0 : \text{ess sup}\, |f| < \infty\right\},$$

and using our fixed section we can define those spaces for any κ-density

$$L^p\left(M, \Omega^\kappa\right) := \left\{\lambda \in \Omega^\kappa : \left(\int_M \left|\lambda |dx|^{\frac{1}{p}-\kappa}\right|^p\right)^{1/p} < \infty\right\},$$

$$L^\infty\left(M, \Omega^\kappa\right) := \left\{f \in \Omega^\kappa : \text{ess sup}\, \left|f |dx|^{-\kappa}\right| < \infty\right\}.$$

L^p−estimates for Safarov ΨDOs

Now, we define the space BMO for κ-densities. Let g a riemannian metric on M and let d its associated geodesic distance. We denote r_0 the injectivity radius of M. Let

$$B_\epsilon(x) = \{y \in M : d(x,y) < \epsilon\}$$

and

$$|B_\epsilon(x)| = \int_{B_\epsilon(x)} |dx|.$$

Then we define the average of a κ-density λ as

$$\overline{\lambda_\epsilon}(x) = \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} \lambda |dx|^{1-\kappa},$$

note that this is a function $\overline{\lambda_\epsilon} : M \to \mathbb{R}$, i.e. a 0-density.

L^p-estimates for Safarov ΨDOs

Finally, we define the BMO norm of a κ-density λ:

$$||\lambda||_{\mathrm{BMO}} = \sup_{\substack{\epsilon < r_0 \\ x \in M}} \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |\lambda(y)|dy|^{-\kappa} - \overline{\lambda_\epsilon}(x)||dy|,$$

therefore

$$\mathrm{BMO}(M, \Omega^\kappa) = \{\lambda \in \Omega^\kappa : ||\lambda||_{\mathrm{BMO}} < \infty\}.$$

L^p-estimates for Safarov ΨDOs

Theorem

Let $p(x,\xi) \in S^{-\beta}_{\rho,\delta}(\nabla)$ with $0 \leqq \delta < \rho < 1$ and $\beta < n(1-\rho)/2$. Assume that at least one of the following conditions is fulfilled:

1. $\rho > \frac{1}{2}$;
2. the connection ∇ is symmetric and $\rho > \frac{1}{3}$;
3. the connection ∇ is flat.

. Then $p(x,D)$ is bounded from L^p to L^p for

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leqq \frac{\beta}{n(1-\rho)}.$$

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Relation with other global pseudodifferential calculus

In [4] Ruzhansky and Turunnen constructed a global pseudodifferential calculus for compact Lie groups using the representation theory of the group G. In this case, the pseudodifferential operators take the following form

$$Af = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\,\mathrm{Tr}\left(\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)\right).$$

Problem: Relate the pseudodifferential calculus defined by symbols modelled on the geometric phase spaces T^*, with connections, to those defined by symbols modelled on the unitary phase spaces $G\times\hat{G}$.

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¡Thanks for your attention!

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