Subriemannian Geometry: Basic Notions and Examples

1. Lecture

"Singular Integrals on nilpotent Lie groups and related topics"

Summer school, Universität Göttingen

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Outline

1. Motivations and the notion of a subriemannian manifold

2. Horizontal curves, connectivity and geodesics

3. Some examples and constructions

Motivation: Subriemannian geometry

Consider n classical particles with coordinates $\{q_1, \cdots, q_n\}$.

Motion under constraints

H: $f(q_1, \cdots, q_n) = 0$, *(holonomic)*,

NH: $f(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) = 0$, *(non-holonomic)*.

Exampels:

H: A particle moving along a surface, or a pendulum.

NH: Rolling of a ball on a plane (or some surface) without slipping or twisting.

Corresponding geometric structures on a manifold

- *holonomic constraints* $\longrightarrow$ integrable distribution (foliation),
- *non-holonomic constraints* $\longrightarrow$ subriemannian manifold.

Some Motivation

A (standard) car cannot move perpendicular to the direction of travel. The process of parking in between two other cars requires maneuvering:

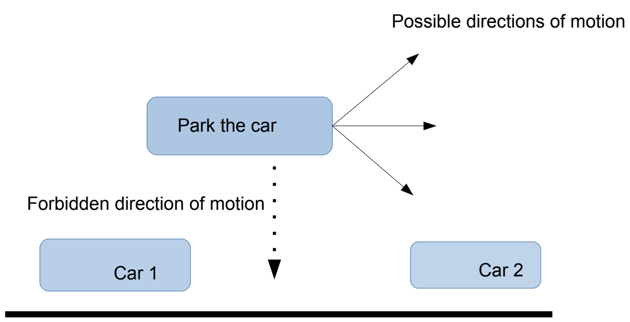
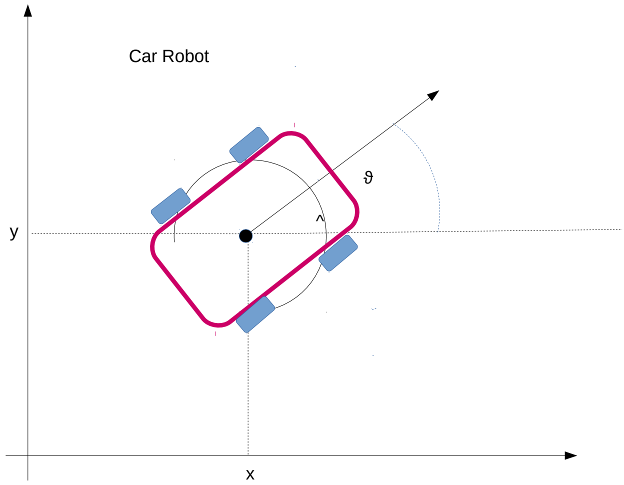


Figure: Parking a car

Next: To formalize the problem we consider the car robot which moves by roto-translation.

Parking a car: roto-translation



Position of the car robot in 3-space: $(x, y, \vartheta) \in \mathbb{R}^2 \times \mathbb{S}^1$.

Possible movements

- $X = \cos\vartheta \cdot \partial_x + \sin\vartheta \cdot \partial_y$, (in direction of the car)
- $Y = \partial_\vartheta$, (rotation)
- $Z = -\sin\vartheta \cdot \partial_x + \cos\vartheta \cdot \partial_y$, (orthogonal to the car).

Parkin a car: roto-translation

Connecting positions: *Which movements allow to reach from any initial position of the car any final position?*

Observations

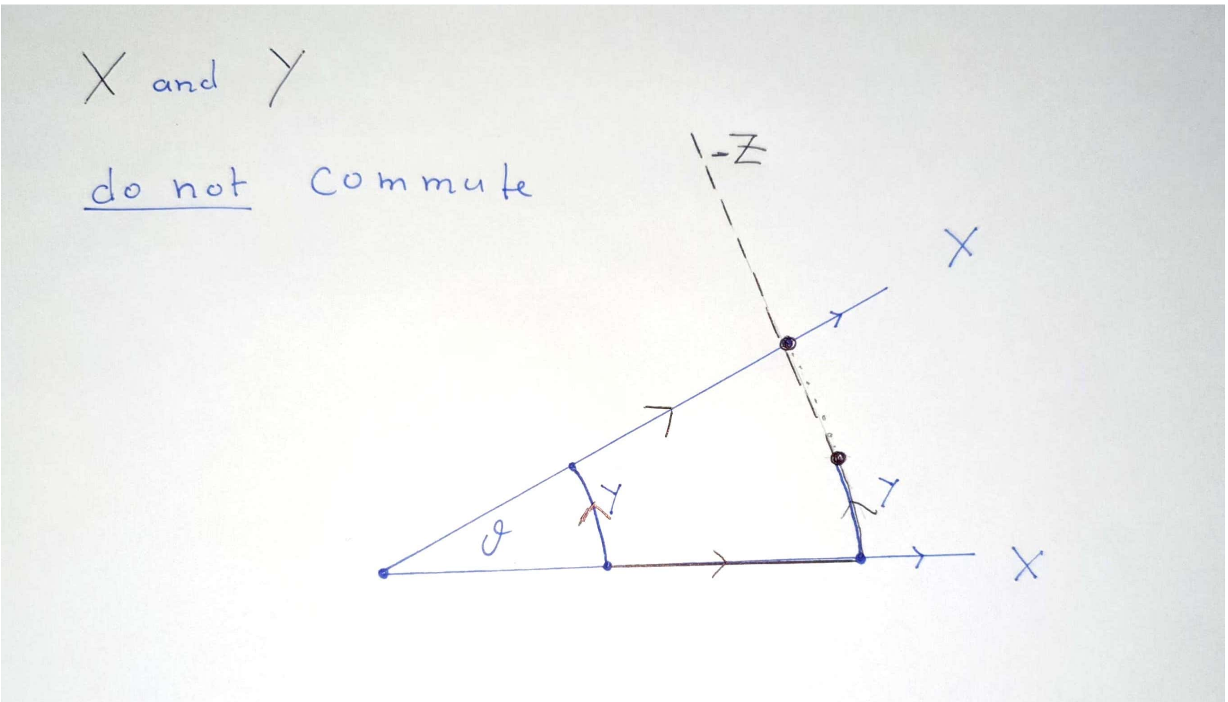
- Moving only along X and Z is not enough: it keeps the angle ϑ fixed.

$$\text{span}\Big\{X, Z\Big\} = \text{kern}\, d\vartheta \quad \textit{and} \quad d\vartheta = \textit{closed form},$$
$$[X, Z] = 0.$$

- Moving along X and Y *(parking procedure)* might be sufficient for connecting positions.

$$\text{span}\Big\{X, Y\Big\} = \text{kern}\,\omega \quad \textit{where} \quad \omega = -\sin\vartheta dx + \cos\vartheta dy.$$
$$[X, Y] = \Big[\cos\vartheta \cdot \partial_x + \sin\vartheta \cdot \partial_y, \partial_\vartheta\Big]$$
$$= -\sin\vartheta \cdot \partial_x + \cos\vartheta \cdot \partial_y = Z.$$

Some Motivation: car robot



Subriemannian Geometry

"Subriemannian geometry models motions under non-holonomic constraints".

Definition

A Subriemannian manifold (shortly: SR-m) is a triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ with:

- M is a smooth manifold (without boundary), $\dim M \geq 3$ and $\mathcal{H} \subset TM$ is a vector distribution.
- $\mathcal{H}$ is bracket generating of rank $k < \dim M$, i.e.

$$\mathrm{Lie}_x \mathcal{H} = T_x M.$$

- $\langle\cdot,\cdot\rangle_x$ is a smoothly varying family of inner products on $\mathcal{H}_x$ for $x \in M$.

1. Example: Heisenberg group

Consider the 3- dimensional Heisenberg group $\mathbb{H}_3 \cong (\mathbb{R}^3, *)$ with product:

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = \Big(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}[x_1 y_2 - y_1 x_2]\Big).$$

Lie algebra of $\mathbb{H}_3$:

On $\mathbb{H}_3 \cong \mathbb{R}^3$ define left-invariant vector fields: Let $q = (x, y, z) \in \mathbb{H}_3$: [1]

$$\begin{aligned}[X_1 f](q) &= \frac{\mathrm{d}f}{\mathrm{d}t}_{|t=0}\Big(q * (t, 0, 0)\Big) \\ &= \frac{\mathrm{d}f}{\mathrm{d}t}_{|t=0}\Big(x + t, 0, z - \frac{yt}{2}\Big) = \Big[\Big(\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}\Big) f\Big](q).\end{aligned}$$

Similarly, with curves $(0, t, 0)_t$ and $(0, 0, t)_t$:

$$X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \quad \text{and} \quad Z = \frac{\partial}{\partial z}.$$

[1] "X left-invariant": $X_{g*h} = (L_g)_* X_h$ with the left-multiplication $L_g : \mathbb{H}_3 \to \mathbb{H}_3$.

Heisenberg group as SR-manifold

Known fact:

The Lie algebra $(\mathfrak{h}_3, [\cdot, \cdot])$ of $\mathbb{H}_3$ can be identified with:

$$\mathfrak{h}_3 = \operatorname{span}\Big\{X_1, X_2, Z\Big\} \quad \textit{with} \quad [\cdot, \cdot] = \text{commtator of vector fields}.$$

Observation

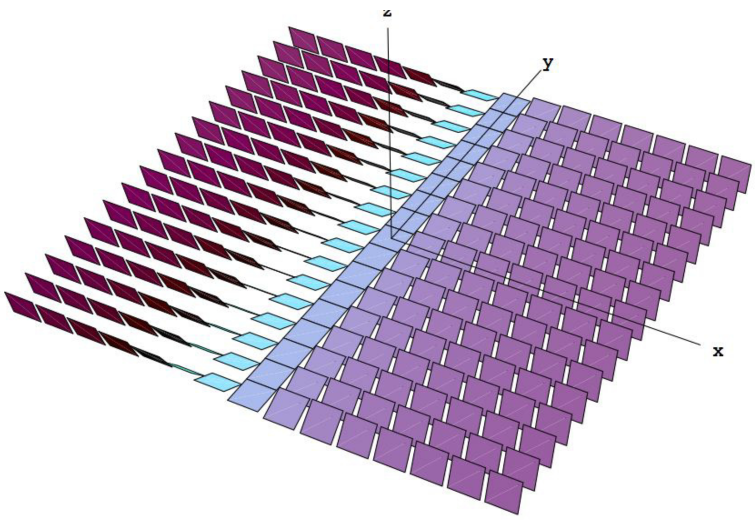
We calculate Lie-brackets $[\cdot, \cdot]$. There is only one non-trivial Lie bracket relation:

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 = Z.$$

- Put $\mathcal{H} = \operatorname{span}\{X_1, X_2\} \subset T\mathbb{H}_3$ (distribution),
- Define $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ by declaring X_1 and X_2 to be pointwise orthonormal.

Conclusion: $(\mathbb{H}_3, \mathcal{H}, \langle \cdot, \cdot \rangle)$ defines a Subriemannian structure on $\mathbb{H}_3$.

Heisenberg group: moving planes



Horizontal curves and cc-distance:

On a SR-manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ we consider horizontal objects, i.e. objects under non-holonomic constraints.

Example

Consider a curve $\gamma : [0, 1] \to M$: [a]

- γ is called horizontal, (a.e.) if it is tangential to $\mathcal{H}$, i.e.

$$\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}.$$

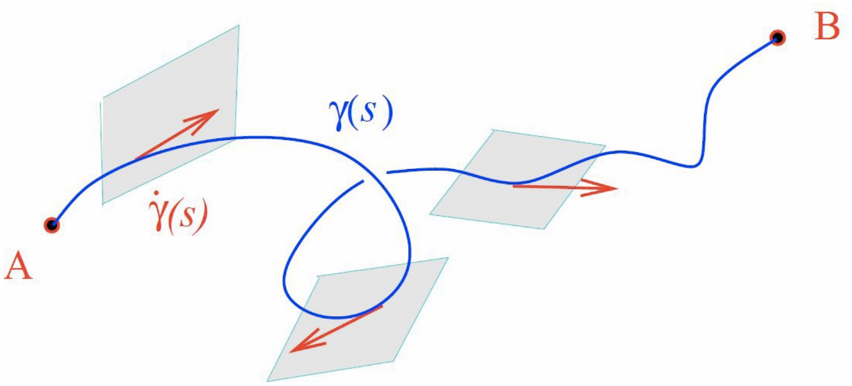
- The curve length is defined by:

$$\ell(\gamma) := \int_0^1 \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} dt.$$

- SR geodesic = locally length minimizing horizontal curve.

[a] piecewise C^1 or just absolutely continuous

Horizontal curves



Carnot-Carathéodory metric

Definition: Sub-Riemannian distanced (cc-distance)

The SR distance between two points $a, b \in M$ is defined by:

$$d_{cc}(a, b) := \inf \Big\{ \ell(\gamma) \ : \ \gamma \ \textit{horizontal} \ , \gamma(0) = a, \gamma(1) = b \Big\}.$$

Question: *Let M be a connected SR-manifold. Can we connect any two points on M by horizontal curves?*

Theorem (W.-L. Chow 1939, P.-K. Rashevskii 1938)

Any two points on a connected SR-manifold can be connected by piecewise smooth horizontal curves.

Geodesic equations

Consequence: The cc-distance d_{cc} [2] on a connected SR-manifold is finite. Hence:

Lemma: *The SR manifold (M, d_{cc}) inherits the structure of a metric space.*

Recall: SR geodesic = locally length minimizing horizontal curve.

Some question:

- How can we obtain Subriemannian geodesics?
- Relation to d_{cc}: can we realize the CC-distance between two point by a (piecewise) smooth SR geodesic?
- Is the distance $x \mapsto d_{cc}(x_0, x)$ smooth for fixed points $x_0 \in M$?

[2] Carnot-Carathéodory distance

Subriemannian geodesics on the Heisenberg group $\mathbb{H}_3$

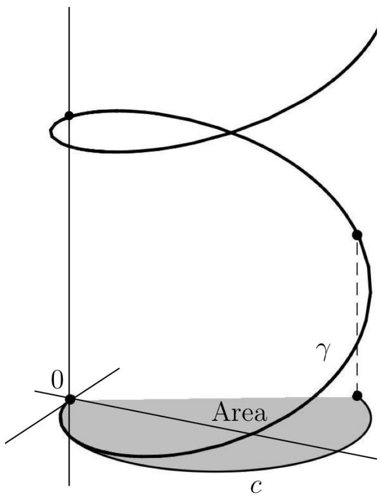


Figure: SR geodesic on $\mathbb{H}_3$ and isoperimetric problem in the plane.

Geodesic equations

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a SR-manifold. Let

$$\big[X_1, \cdots, X_m\big] = \textit{vector fields} \quad \textit{and} \quad m = \text{rank}\, \mathcal{H}$$

be an local orthonormal frame around a point $q \in M$, i.e.

$$\mathcal{H}_q = \text{span}\Big\{X_1(q), \cdots, X_m(q)\Big\} \quad \textit{and} \quad \big\langle X_i(q), X_j(q)\big\rangle = \delta_{ij}.$$

Idea: *Expand locally the derivative of a horizontal curve with respect to the above frame*

SR-geodesics and optimal control

Observation

Let $\gamma : [0, 1] \to M$ be horizontal. With suitable coefficients $u_i(t)$ one can write

$$\gamma'(t) = \sum_{j=1}^{m} u_j(t) \cdot X_j(t) \quad \Longrightarrow \quad \big\langle \gamma'(t), \gamma'(t) \big\rangle = \sum_{j=1}^{m} u_i^2(t).$$

Finding SR-geodesics between $A, B \in M$= optimal control problem OCP.

OCP: *Minimize the cost*

$$J_T(u) := \frac{1}{2} \int_0^T \sqrt{\sum_{j=1}^{m} u_i^2(t) dt}$$

under the conditions

$$\gamma' = \sum_{j=1}^{m} u_j \cdot X_j(\gamma) \quad \textit{and} \quad \gamma(0) = A, \ \gamma(T) = B.$$

SR-geodesic: a Hamiltonian formalism

Remark:

Instead of minimizing a length we may equivalently minimize an "energy":

OCP: *Minimize the cost*

$$J_T(u) := \frac{1}{2}\int_0^T \sum_{j=1}^m u_i^2(t)dt$$

under the conditions

$$\gamma' = \sum_{j=1}^m u_j \cdot X_j(\gamma) \quad \text{and} \quad \gamma(0) = A,\ \gamma(T) = B.$$

Hamiltonian formalism (as known in Riemannian geometry):

Assign a Subriemannian Hamiltonian $H_{sr} \in C^\infty(T^*M)$ to the problem:

$$H_{sr}(q,p) = \sum_{j=1}^m p\Big(X_j(q)\Big)^2, \quad \text{where} \quad (q,p) \in T_q^*M.$$

SR-geodesic: a Hamiltonian formalism

With the Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(T^*M)$ consider:

$$\vec{H}_{sr} = \{\cdot, H_{sr}\} = \frac{\partial H_{sr}}{\partial p}\cdot\frac{\partial}{\partial q} - \frac{\partial H_{sr}}{\partial q}\cdot\frac{\partial}{\partial p} = \text{Hamiltonian vector field.}$$

The Hamiltonian vector field defines the geodesic flow on T^*M and projections of the flow to M give SR-geodesics:

Theorem (normal geodesics)

Let $\zeta(t) = (\gamma(t), p(t))$ be a solution to the normal geodesic equations:

$$\dot{q}_i = \frac{\partial H_{sr}}{\partial p_i}(q,p) \quad \text{and} \quad \dot{p}_i = -\frac{\partial H_{sr}}{\partial q_i}(q,p), \quad i = 1,\cdots,\dim M.$$

Then $\gamma(t)$ locally minimizes the SR-distance.

Proof: [3]

[3] R. Montgomery, *A tour of Subriemannian Geometries, Their Geodesics and Applications* Math. Surveys and Monographs, 2002.

SR-geodesics

Remark

There are various differences to the setting of a Riemannian manifold:

- The Hamiltonian in Riemannian geometry can be expressed as

$$H_R(q,p) = \sum_{i,j=1}^{n} g^{ij}(q) p_i p_j, \qquad g^{ij} := \textit{inverse metric tensor.}$$

  In SR-geometry g_{ij} is an $m \times m$-matrix and not invertible.

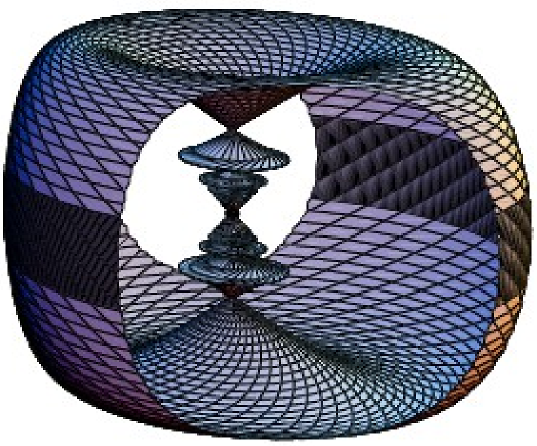
- There are no 2nd order geodesic equations in the SR-setting such as

$$\ddot{q}^k = \Gamma^k_{ij} \dot{q}_i \dot{q}_j \qquad \textit{or shortly:} \qquad \nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

  The obtained regularity of SR-geodesics is not clear.

- In SR-geometry there may be singular geodesics which **do not solve** the geodesic equations in the above theorem.

Heisenberg group $\mathbb{H}_3$: sphere in d_{cc}-metric.



Front of SR geodesics at time T
(picture by: U. Boscain, D. Barilari)

Examples of SR manifolds

Lie groups: A Lie group G has trivial tangent bundle and the last construction of a trivial bundle can be generalized:

Left-invariant structure

- Let $\mathfrak{g}$ denote the Lie algebra of G.
- Let $V \subset \mathfrak{g}$ be a subspace of $\mathfrak{g}$ with inner product $\langle \cdot, \cdot \rangle_V$ and

$$\mathfrak{g} = \text{Lie}(V) = \text{span}\Big\{ v, [w,x], \big[y,[w,x]\big], \cdots \,:\, x,y,w \in V \Big\}.$$

  Identify V (via left-translation) with a space of left-invariant vector fields on G.
- The G becomes a Subriemannian manifold $(G, \mathcal{H}, \langle \cdot, \cdot \rangle)$ with:

$$\mathcal{H} = V$$
$$\langle \cdot, \cdot \rangle_q = \langle (dL_q)^{-1} \cdot, (dL_q)^{-1} \cdot \rangle_V.$$

Examples of SR manifolds

Contact structures Let Θ be a one-form on a manifold M of dimension $\dim M = 2k+1$. Put:

$$\mathcal{H}_q := \text{kern}(\Theta_q) \subset T_q M, \qquad (q \in M).$$

Properties

- the restriction of $d\Theta_q$ to $\mathcal{H}_q$ is non-degenerate [a] for each $q \in M$:

  If $v \in \mathcal{H}$ with $d\Theta(v,w) = 0$ for all $w \in \mathcal{H}_q$, then $v = 0$.

- **equivalently:** the form

$$\omega := \Theta \wedge \big(d\Theta\big)^{2k} \neq 0$$

  does not vanish at any point of M ($= \omega$ is a volume form):

[a] a symplectic form

Contact manifolds

Lemma

Let Θ be a contact form on M. Then

$$\mathcal{H} := \ker \Theta \subset TM$$

is a bracket generating distribution.

Proof: Use Cartan's formula: With vector fields X, Y on M:

$$d\Theta(X, Y) = X\Theta(Y) - Y\Theta(X) - \Theta([X, Y]).$$

Let X, Y be horizontal, i.e. $X_q, Y_q \in \mathcal{H}_q = \text{kern}\, \Theta_q$ for all $q \in M$. Then

$$\Theta(X) = \Theta(Y) = 0 \quad \Longrightarrow \quad d\Theta(X, Y) = -\Theta\big([X, Y]\big).$$

Since $d\Theta$ is non-degenerate on $\mathcal{H}_q$ we find X, Y with

$$[X, Y]_q \notin \text{kern}\Theta_q = \mathcal{H}_q.$$

$\square$

Contact manifolds (continued)

Choose an almost complex structure $J : \mathcal{H} \to \mathcal{H}$ such that

$$\langle \cdot, \cdot \rangle = d\Theta\big(J\cdot, \cdot\big), \qquad \textit{and} \qquad J^2 = -\text{I}$$

is an inner product on $\mathcal{H}$ (symmetric, positive definite).

Definition (Contact Subriemannian manifold)

The tripel $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ is called contact Subriemannian manifold.

Example: Consider again the Heisenberg group $\mathbb{H}_3 \cong \mathbb{R}^3$ with distribution:

$$\mathcal{H} = \text{span}\Big\{ \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \Big\} = \text{kern}\Big(\underbrace{dz - \frac{x}{2}dy + \frac{y}{2}dx}_{=\Theta} \Big).$$

Moreover, Θ is a contact form and $\mathbb{H}_3$ is a contact SR-manifold:

$$\Theta \wedge d\Theta = -\Theta \wedge \big(dx \wedge dy\big) = -dx \wedge dy \wedge dz \neq 0.$$

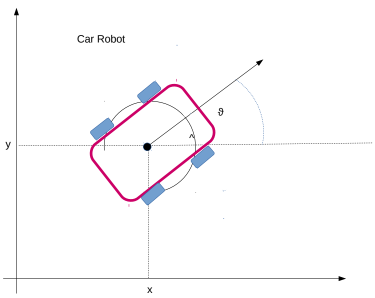
Roto-translation group: How to park a car?

Possible movements

- $X = \cos\vartheta \cdot \partial_x + \sin\vartheta \cdot \partial_y$, (in direction of the car)
- $Y = \partial_\vartheta$, (rotation)
- $Z = -\sin\vartheta \cdot \partial_x + \cos\vartheta \cdot \partial_y$, (orthogonal to the car).

Good choice:

$$\mathcal{H} = \text{span}\Big\{X, Y\Big\} = \text{kern}\,\omega \quad \textit{with} \quad \omega = -\sin\vartheta \cdot dx + \cos\vartheta \cdot dy.$$



$$\omega \wedge d\omega = \omega \wedge \big(-\cos\vartheta \cdot d\vartheta \wedge dx - \sin\vartheta \cdot d\vartheta \wedge dy\big) = -dx \wedge dy \wedge d\vartheta \neq 0.$$

Subriemannian structures of bundle type

Let (M, g_M) and (N, g_N) be Riemannian manifolds with Riemannian submersion:

$$\pi : M \to N.$$

Properties

Let $q \in M$ and $p = \pi(q) \in N$.

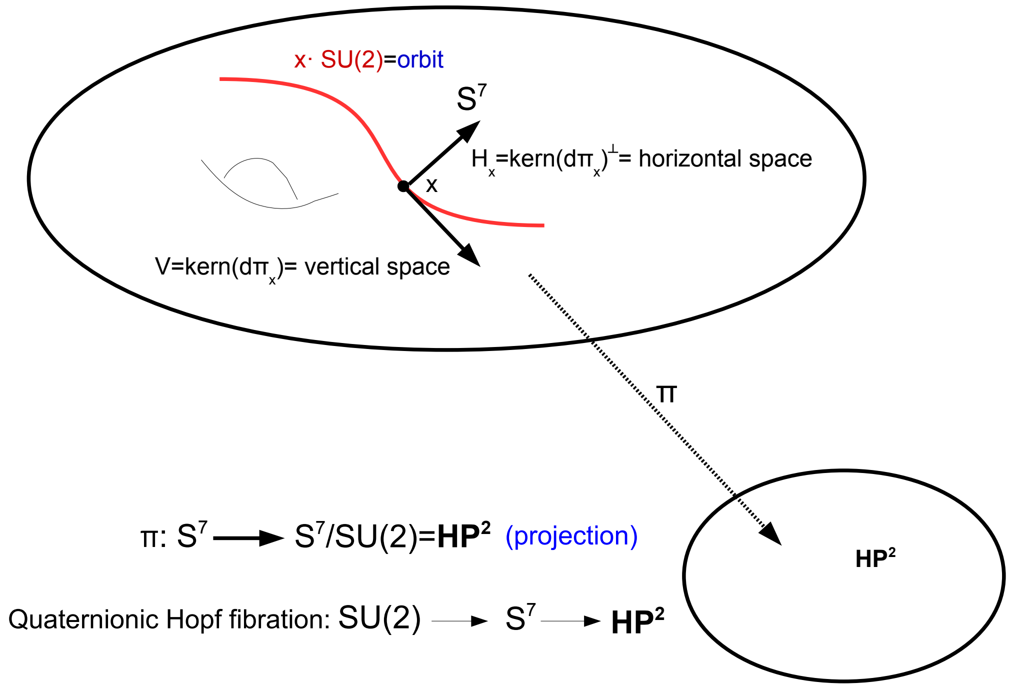
- kern $d\pi_q \subset T_qM$ is a the space tangent to the fiber $\pi^{-1}(p)$ at q.
- The restriction of the differential

$$d\pi_q : \mathcal{H}_q := \big(\text{kern}\, d\pi_q\big)^\perp \subset T_qM \to T_pN$$

  is an isometry.
- On $\mathcal{H}$ consider the restriction $\langle \cdot, \cdot \rangle$ of the metric on TM

These data may give a SR-structure of bundle type. (Note: bracket generating property is not clear in general and has to be checked).

Example: Quaternionic Hopf fibration



Example: Hopf fibration

Consider the three sphere as a subset of $\mathbb{C}^2$:

$$\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2.$$

Definition (Hopf fibration)

The Hopf fibration is the submersion map

$$\pi : \mathbb{S}^3 \to \mathbb{S}^2_{\frac{1}{2}} : \pi(z) := \frac{1}{2}\Big(|z_1|^2 - |z_2|^2, \mathrm{Re}(z_1\bar{z}_2), \mathrm{Im}(z_1\bar{z}_2)\Big),$$

where $\mathbb{S}^2_{\frac{1}{2}}$ is the 2-sphere of radius $1/2$.

Theorem: *The Hopf fibration defines a principal $\mathbb{S}^1$-bundle, where $\mathbb{S}^1$ acts by componentwise multiplication on $\mathbb{S}^3 \subset \mathbb{C}^2$.*

Remark: The corresponding distribution on $\mathbb{S}^3$ of bundle type is bracket generating (and coincides with a contact structure on $\mathbb{S}^3$).

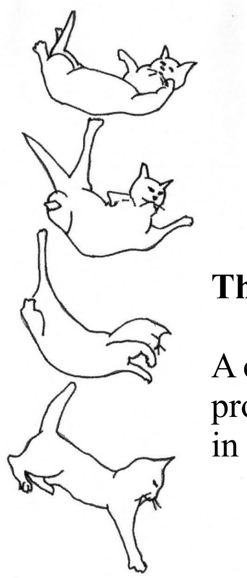
"Why does it matter?"

Concepts of SR geometry have been around for a long time and play a role in mathematics, physics or applied sciences:

Applications in:

- classical mechanics, quantum mechanics, thermodynamics, quantum computing
- control theory
- geometric structures and classifications
- rolling of manifolds, falling cat problem, parking a car $\cdots$
- vision theory
- image reconstruction via hypoelliptic diffusion
- PDE, analysis of hypoelliptic operators $\longrightarrow$ *this course*

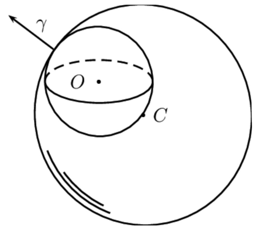
Example 1: The falling cat problem and rolling manifolds



The falling cat:

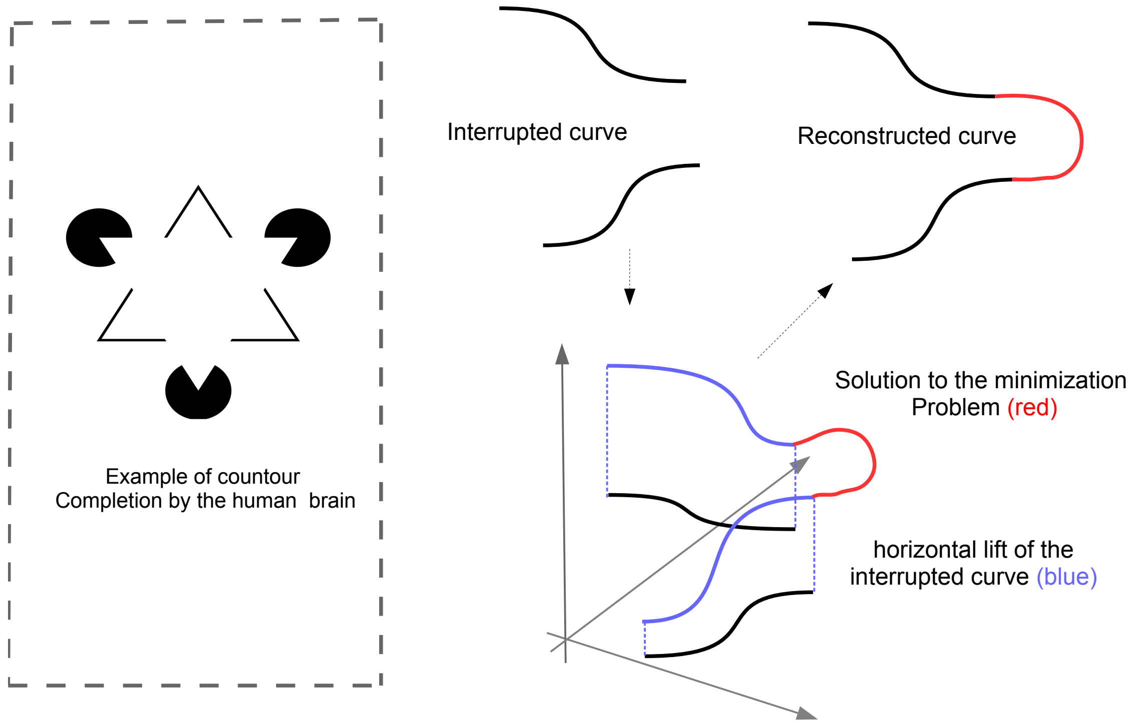
A connectivity problem in SR geometry

rolling sphere



Example 2: curve reconstruction

How does the brain reconstruct an interrupted curve?



Summary

- SR geometry models motion under non-holonomic constraints:
  - mechanical systems,
  - rolling of manifolds,
  - parking a car,
  - falling cat$\cdots$)
- Connected SR-manifolds are metric spaces with the CC-distance.
- Subriemannian geodesics $\longleftrightarrow$ optimal control problem. (Quite different behavior in comparison with geodesics in Riemannian geometry.)
- Examples include: some Lie groups, (e.g. Heisenberg group or $\mathbb{S}^3$), Euclidean spheres, some principal bundles (e.g. Hopf fibration), H-type foliations,... *and much more.*
- SR geometry naturally appears in a wide range of problems and has many applications (not only inside mathematics).

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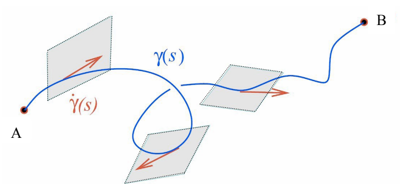
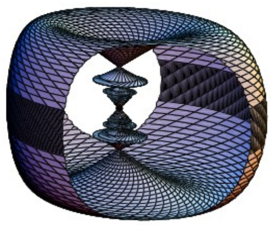
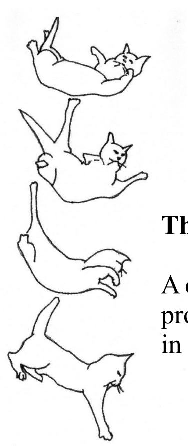
Thank you for your attention!



Distribution and horizontal curve



Front of SR geodesics at time T
(picture by: U. Boscain, D. Barilari)



The falling cat:

A connectivity problem in SR geometry