Popp measure and the intrinsic Sub-Laplacian

2. lecture

"Singular Integrals on nilpotent Lie groups and related topics"

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Outline

1. From the Riemannian to the Subriemannian Laplacian
2. Nilpotentization and Popp measure
3. Subriemannian isometries
4. Sub-Laplacian on nilpotent Lie groups

Subriemannian Geometry (Reminder from the 1. talk)

"Subriemannian geometry models motions under non-holonomic constraints".

Definition

A Subriemannian manifold (shortly: SR-m) is a triple $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ with:

- M is a smooth manifold (without boundary), $\dim M \geq 3$ and $\mathcal{H} \subset TM$ is a vektor distribution.
- $\mathcal{H}$ is bracket generating of rank $k < \dim M$, i.e.

$$\mathrm{Lie}_x \mathcal{H} = T_x M$$

- $\langle \cdot, \cdot \rangle_x$ is a smoothly varying family of inner products on $\mathcal{H}_x$ for $x \in M$.

Question: *Can we assign "geometric operators" to such a structure similar to the Laplacian in Riemannian geometry?*

Regular Distribution

Let $\mathcal{H} \subset TM$ denote a distribution on M. Define vector spaces depending on $q \in M$:

$$\mathcal{H}^1 := \mathcal{H}, \quad \text{and} \quad \mathcal{H}^{r+1} := \mathcal{H}^r + \big[\mathcal{H}^r, \mathcal{H}\big],$$

where

$$\big[\mathcal{H}^r, \mathcal{H}\big]_q := \mathrm{span}\Big\{ \big[X, Y\big]_q \; : \; X \in \mathcal{H}^r \; \text{and} \; Y \in \mathcal{H} \Big\}.$$

This gives a flag of vector spaces

$$\mathcal{H}_q = \mathcal{H}_q^1 \subset \mathcal{H}_q^2 \subset \cdots \subset \mathcal{H}_q^r \subset \mathcal{H}_q^{r+1} \subset \cdots T_q M$$

Remark: $\mathcal{H}$ bracket generating: $\forall \, q \in M$, $\exists \, \ell_q \in \mathbb{N}$ with $\mathcal{H}_q^{\ell_q} = T_q M$.

Definition

$\mathcal{H}$ is called regular, if the dimensions $\dim \mathcal{H}_q^r$ are independent of $q \in M$.

Ex: The 3-dimensional Heisenberg group $\mathbb{H}_3$ is regular as a SR manifold.

A non-regular distribution, (Martinet distribution)

Example: A distribution which is not regular across a line:

On $M = \mathbb{R}^3$ with coordinates $q = (x, y, z)$ consider the vector fields:

$$X := \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z} \quad \textit{and} \quad Y := \frac{\partial}{\partial y}.$$

Then we have the Lie bracket:

$$[X, Y] = -y\frac{\partial}{\partial z}.$$

Therefore:

$$\mathcal{H}_q^2 = \text{span}\Big\{X, Y, y\frac{\partial}{\partial z}\Big\} \quad \Longrightarrow \quad \mathcal{H}_{(x,0,z)}^2 = \text{span}\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big\}.$$

Observe: The dimension of $\mathcal{H}_q^2$ "jumps" at a hyperplane:

$$\dim \mathcal{H}_q^2 = \begin{cases} 2, & \text{if } y = 0, \\ 3, & \text{if } y \neq 0. \end{cases}$$

From Riemannian to the Subriemannian Laplacian

Goal: *In analogy to the Laplace operator in Riemannian geometry we want to assign a Sub-Laplace operator to the Subriemannian structure.*

1. Recall the definition of the Beltrami-Laplace operator:

Let (M, g) be an oriented Riemannian manifold with $\dim M = n$ and let $[X_1, \cdots, X_n]$ be a local orthonormal frame around a point $q \in M$.

Definition

The Riemannian volume form ω is defined through the requirement:

$$\omega(X_1, \cdots, X_n) = 1.$$

Or in coordinates:

$$\omega = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n \quad \textit{where} \quad g_{ij} = g(\partial_i, \partial_j), \;\; \partial_i := \frac{\partial}{\partial x_i}.$$

Define the Beltrami-Laplace operator by: $\Delta = \text{div}_\omega \circ \text{grad}$.

The Sub-Laplacian

Analogously to the definition of the Laplace operator we assign a differential operator to a SR- manifold with regular distribution $\mathcal{H}$.

We need two ingredients, namely:

- a *SR gradient*,
- a *SR divergence*.

Horizontal gradient and ω-divergence

Let ω be a smooth measure, X a vector field on M and $\varphi \in C^\infty(M)$: [a]

$$\mathcal{L}_X(\omega) = \mathrm{div}_\omega(X)\,\omega \qquad (\omega\text{-divergence})$$

$$\Big\langle \underbrace{\mathrm{grad}_{\mathcal{H}}(\varphi)}_{\in \mathcal{H}_q}, v \Big\rangle_q = d\varphi(v), \quad v \in \mathcal{H}_q \qquad (\text{horizontal gradient}).$$

These equations - together with the horizontality condition of the gradient - define $\mathrm{div}_\omega(X)$ and $\mathrm{grad}_{\mathcal{H}}(\varphi)$.

[a] $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$.

Sub-Laplacian

Definition

The Sub-Laplacian on a SR-manifold $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ associated to a smooth volume ω is defined by

$$\Delta_{\mathrm{sub}} := \mathrm{div}_\omega \circ \mathrm{grad}_{\mathcal{H}}.$$

Consider a local orthonormal frame for $\mathcal{H}$

$$[X_1, \cdots, X_m] \qquad \textit{with} \qquad m \leq n = \dim M.$$

Similar to the Laplacian we can express Δ_{sub} in the form:

$$\Delta_{\mathrm{sub}} = \sum_{i=1}^m \Big[X_i^2 + \mathrm{div}_\omega(X_i) \cdot X_i \Big].$$

The Sub-Laplacian

Theorem

The SR Laplacian Δ_{sub} associated to a smooth measure ω is negative, symmetric and, if M is compact, essentially self-adjoint on the space $C_c^\infty(M) \subset L^2(M)$.

Proof: (1. statement) Let $f \in C_c^\infty(M)$ and let X be a vector field on M. One shows:

$$\int_M f \cdot \underbrace{\text{div}_\omega(X)\,\omega}_{=\mathcal{L}_X(\omega)} = -\int_M X(f)\,\omega = -\int_M df(X)\,\omega.$$

Choose $X = \text{grad}_{\mathcal{H}}(g)$ with $g \in C_c^\infty(M)$. Symmetry and negativity follow:

$$\int_M f \cdot (\Delta_{\text{sub}} g)\,\omega = -\int_M \langle \text{grad}_{\mathcal{H}} f, \text{grad}_{\mathcal{H}} g\rangle \omega.$$

Essentially selfadjointness needs more work. □

Nilpotentization and Popp measure

Question: *How to choose the smooth measure ω in the ω-divergence when defining the sub-Laplacian*

$$\Delta_{\text{sub}} := \text{div}_\omega \circ \text{grad}_{\mathcal{H}}?$$

Assumption: Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a regular subriemannian manifold.

Consider again the flag induced by the bracket generating distribution $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^r \subset \mathcal{H}^{r+1} \subset \cdots$$

Reminder: $\dim \mathcal{H}_q^r$ for all r are independent of $q \in M$, where:

$$\mathcal{H}^1 := \mathcal{H} = \text{"sheave of smooth horizontal vector fields"},$$
$$\mathcal{H}^{r+1} := \mathcal{H}^r + [\mathcal{H}^r, \mathcal{H}],$$

with

$$[\mathcal{H}^r, \mathcal{H}]_q = \text{span}\Big\{[X, Y]_q \,:\, X \in \mathcal{H}^r \text{ and } Y \in \mathcal{H}\Big\}.$$

Nilpotentization

For each $q \in M$ we obtain a direct sum of vector spaces:

$$\begin{aligned} \mathrm{gr}(\mathcal{H})_q &= \mathcal{H}_q \oplus \mathcal{H}_q^2/\mathcal{H}_q \oplus \cdots \oplus \mathcal{H}_q^r/\mathcal{H}_q^{r-1} \\ &= \text{nilpotentization.} \end{aligned}$$

Observations:

(a) Lie brackets of vector fields on the manifold M induce a Lie algebra structure on $\mathrm{gr}(\mathcal{H})_q$ such that

$$v \in \mathcal{H} \text{ and } w \in \mathcal{H}_q^j/\mathcal{H}_q^{j-1} \quad \Longrightarrow \quad [v, w] \in \mathcal{H}_q^{i+1}/\mathcal{H}_q^i.$$

(b) The Lie algebra in (a) is nilpotent, i.e. there is $n \in \mathbb{N}$ such that:

$$\Big[X_1\big[X_2 \cdots \big[X_n, X\big]\cdots\big]\Big] = 0, \qquad \forall\ X_1, \cdots, X_n, X \in \mathfrak{g}. \qquad (*)$$

The minimal n in (b) is called the step of the nilpotent Lie algebra.

Example: *The step of the nilpotentization* $\mathrm{gr}(\mathcal{H})_q$ *is* r*.*

Popp measure: construction in the case $r = 2$

$(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ = regular SR manifold. Let $r = 2$ and $q \in M$.

1. step: Let $v, w \in \mathcal{H}_q$ and V, W be horizontal vector fields near p with:

$$V(q) = v \quad \text{and} \quad W(q) = w.$$

Consider the map

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := \big[V, W\big]_q \bmod \mathcal{H}_q.$$

Some properties of π:

- π is surjective,
- the inner product on $\mathcal{H}_q$ induces an inner product on the Hilbert space tensor product $\mathcal{H}_q \otimes \mathcal{H}_q$.

Question: *Is the map* π *well-defined?*

With horizontal vector fields V and W with $V_q = v$ and $W_q = w$:

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := \big[V, W\big]_q \text{ mod } \mathcal{H}_q.$$

Lemma

The map π is independent of the choice of V and W.

Proof: Let $\widetilde{V}$ and $\widetilde{W}$ be different horizontal extensions of v and w, i.e.

$$\widetilde{V}_p, \widetilde{W}_p \in \mathcal{H}_p \quad \forall\, p \in U_q, \qquad \textit{and} \qquad \widetilde{V}_q = v, \;\; \widetilde{W}_q = w.$$

With a local frame $[X_1, \cdots, X_m]$ of $\mathcal{H}$ we can write:

$$\widetilde{V} = V + \sum_{i=1}^{m} f_i X_i \qquad \textit{and} \qquad \widetilde{W} = W + \sum_{i=1}^{m} g_i X_i,$$

where $f_i, g_i \in C^\infty(M)$ fulfill $f_i(q) = g_i(q) = 0$.

Proof: (continued)

We form Lie brackets:

$$\big[\widetilde{V}, \widetilde{W}\big] = \Big[V + \sum_{i=1}^{m} f_i X_i, W + \sum_{j=1}^{m} g_j X_j\Big]$$

$$= [V, W] + \sum_{j=1}^{m} \big[V, g_j X_j\big] - \big[W, f_j X_j\big] + \sum_{i,j=1}^{m} \big[f_i X_i, g_j X_j\big] = (*).$$

Use the rule:

$$\big[V, g_j X_j\big] = V(g_j) X_j + g_j \big[V, X_j\big]$$

to obtain $[f_i X_i, g_j X_j]_q = f_i(q) X_i(g_j)_q (X_j)_q + g_j(q) [f_i X_i, X_j]_q = 0$ and

$$\big[\widetilde{V}, \widetilde{W}\big]_q = [V, W]_q + \sum_{j=1}^{m} \Big(V(g_j) - W(f_j)\Big)(X_j)_q = \big[V, W\big]_q \text{ mod } \mathcal{H}_q.$$

This proves the statement. $\square$

Summary

Consider the map

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := \big[V, W\big]_q \operatorname{mod} \mathcal{H}_q.$$

Some properties of π:

- π is surjective,
- the inner product on $\mathcal{H}_q$ induces an inner product on $\mathcal{H}_q \otimes \mathcal{H}_q$.
- Finally:

$$\mathcal{H}_q^2/\mathcal{H}_q \cong \operatorname{kern}(\pi)^\perp \subset \mathcal{H}_q \otimes \mathcal{H}_q.$$

Consequence: The inner product on $\mathcal{H}_q$ induces a inner product on:

$$\operatorname{gr}(\mathcal{H})_q = \mathcal{H}_q \oplus \mathcal{H}_q^2/\mathcal{H}_q = \textit{nilpotentization}.$$

This inner product induces a canonical volume form, i.e. an element [1]

$$\mu_q \in \Lambda^n \operatorname{gr}(\mathcal{H})_q^* \cong \big(\Lambda^n \operatorname{gr}(\mathcal{H})_q\big)^*.$$

[1](”Wedge” elements of an orthonormal basis).

Definition: Popp measure ($r = 2$)

2. step: *We need to produce a volume form on M itself.*

Let $n = \dim M$. Then there is a canonical isomorphism

$$\Theta_q : \Lambda^n(T_qM) \to \Lambda^n \operatorname{gr}(\mathcal{H})_q.$$

Explicitly:

Let $v_1, \cdots, v_n$ be a basis of T_qM s. t.$v_1, \cdots, v_m$ is a basis of $\mathcal{H}_q$. Put

$$\Theta_q\big(v_1 \wedge \cdots \wedge v_n\big) := v_1 \wedge \cdots \wedge v_m \,\widehat{\otimes}\, \Big(v_{m+1} + \mathcal{H}_q\Big) \wedge \cdots \wedge \Big(v_n + \mathcal{H}_q\Big).$$

Then Θ_q is independent of the choice of such basis.

Definition: Popp measure

With the volume form $\mathcal{P}_q := \Theta_q^*(\mu_q) = \mu_q \circ \Theta_q \in (\Lambda^n T_qM)^*$ we form

$$\mathcal{P} \in \Omega^n(M) = \textit{Popp measure}.$$

Remark: $\mathcal{P}$ *generalizes to SR-structures of arbitrary step* $r > 0$.

Intrinsic Sub-Laplacian

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a regular SR-manifold with Popp measure $\mathcal{P}$.

Definition

The intrinsic Sub-Laplacian on M is the one associated to the Popp measure $\mathcal{P}$:

$$\Delta_{\mathcal{P}} = \mathrm{div}_{\mathcal{P}} \circ \mathrm{grad}_{\mathcal{H}}.$$

1. Example: Martinet distribution

Consider the Martinet distribution on $\mathbb{R}^3$:

Define vector fields:

$$X := \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}, \qquad Y := \frac{\partial}{\partial y} \quad \textit{and} \quad Z = \frac{\partial}{\partial z}.$$

Consider the following distribution: With $q \in \mathbb{R}^3$ put:

$$\begin{aligned} \mathcal{H}_q :&= \mathrm{span}\Big\{X_q, Y_q\Big\} \\ &= \mathrm{kern}\,(\Theta_q) \quad \textit{where} \quad \Theta = dz - \frac{y^2}{2}dx \\ &= \text{Martinet distribution}. \end{aligned}$$

- An inner product on $\mathcal{H}_q$ is defined by declaring X_q and Y_q orthonormal.
- **bracket relations:**

$$\big[X, Y\big] = -yZ \quad \textit{and} \quad \Big[Y, \big[X, Y\big]\Big] = -Z.$$

1. Example: Martinet distribution (continued)

The Martinet distribution $\mathcal{H}$ is

- bracket generating on $\mathbb{R}^3$ and of step 3 if $y = 0$,
- regular of step 2 restricted to

$$M_{y\neq 0} := \{(x, y, z)^t \ : \ y \neq 0\}.$$

Popp measure on $M_{y\neq 0}$: Consider the map:

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := [X, Y]_q \bmod \mathcal{H}_q,$$

where $v = X_q$ and $w = Y_q$. Then:

$$\begin{aligned}(\ker \pi)^\perp &= \operatorname{span}\Big\{X \otimes X, Y \otimes Y, \frac{1}{\sqrt{2}}(X \otimes Y + Y \otimes X)\Big\}^\perp \\ &= \operatorname{span}\Big\{\frac{1}{\sqrt{2}}(X \otimes Y - Y \otimes X)\Big\}.\end{aligned}$$

1. Example: Martinet distribution (continued)

Using $[X, Y] = -yZ$ we find:

$$\begin{aligned}\frac{1}{\sqrt{2}}\pi\Big[X \otimes Y - Y \otimes X\Big] &= \sqrt{2} \cdot [X, Y] + \mathcal{H}_q \\ &= -\sqrt{2}yZ + \mathcal{H}_q.\end{aligned}$$

This induces an inner product norm on $\mathcal{H}_q^2/\mathcal{H}_q = \operatorname{span}\{Z\} + \mathcal{H}_q$ via:

$$\|Z + \mathcal{H}_q\|_q = \frac{1}{\sqrt{2}|y|}.$$

Take the dual basis to $[X, Y, \sqrt{2}|y|Z]$ which is

$$\Big[X^* = dx, Y^* = dy, (\sqrt{2}|y|Z)^* = (\sqrt{2}|y|)^{-1}(dz - \frac{y^2}{2}dx)\Big].$$

Popp measure:

$$\mathcal{P} = X^* \wedge Y^* \wedge (\sqrt{2}|y|Z)^* = \frac{1}{\sqrt{2}|y|} dx \wedge dy \wedge dz.$$

Intrinsic Sub-Laplacian for the Martinet distribution

Knowing the Popp measure, we can calculate the intrinsic Sub-Laplacian

$$\Delta_{\text{sub}} = \text{div}_{\mathcal{P}} \circ \text{grad}_{\mathcal{H}} \quad \textit{on} \quad M_{y\neq 0} \subset \mathbb{R}^3.$$

Recall the following explicit expression:

$$\Delta_{\text{sub}} = X^2 + Y^2 + \text{div}_{\mathcal{P}}(X)\, X + \text{div}_{\mathcal{P}}(Y)\, Y.$$

Note that

$$\text{div}_{\mathcal{P}}(X) \cdot \mathcal{P} = \mathcal{L}_X \mathcal{P} = d\big(\iota_X \mathcal{P}\big) = d\left(\frac{1}{\sqrt{2}|y|} dy \wedge dz\right) = 0 \cdot \mathcal{P},$$

$$\text{div}_{\mathcal{P}}(Y) \cdot \mathcal{P} = \mathcal{L}_Y \mathcal{P} = d\big(\iota_Y \mathcal{P}\big) = -d\left(\frac{1}{\sqrt{2}|y|} dx \wedge dz\right) = -\frac{1}{y} \cdot \mathcal{P}.$$

Intrinsic Sub-Laplacian

The intrinsic Sub-Laplacian becomes singular at the $y = 0$ - surface.

$$\Delta_{\text{sub}} = X^2 + Y^2 - \frac{1}{y}\, Y.$$

Popp measure and local isometries

Riemannian isometry: *diffeomorphism* with differential being an isometry for the Riemannian metric.

Definition (volume preserving transformation)

Let M be a manifold and $\mu \in \Omega^n(M)$ a volume form. A diffeomorphism $\Phi : M \to M$ is a volume preserving transformation if

$$\phi^* \mu = \mu.$$

Standard fact:

Riemannian isometries are volume preserving transformation for the standard Riemannian volume ω.

Question: Is there an analogous statement in the case of a SR manifold and the Popp measure?

Subriemannian isometries

Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a SR manifold and

$$\Phi : M \to M \tag{$*$}$$

a diffeomorphism.

Definition

The map $(*)$ is called isometry, if its differential $\Phi_* : TM \to TM$ preserves the SR structure, i.e.

- $\Phi_*(\mathcal{H}_q) = \mathcal{H}_{\Phi(q)}$ for all $q \in M$,
- For all $q \in M$ and all horizontal vector fields X, Y:

$$\langle \Phi_* X, \Phi_* Y\rangle_{\Phi(q)} = \langle X, Y\rangle_q.$$

We write $\mathrm{Iso}(M)$ for the group of all isometries on the SR manifold M.

Popp volume and isometries

Theorem (D. Barilari, L. Rizzi, 2012)

Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a regular SR manifold.

(a) *SR isometries are volume preserving for Popp's volume.*

(b) *If $\mathrm{Iso}(M)$ acts transitively, then Popp's volume is the unique volume (up to multiplication by a constant) with (a).*

Example

Let $M = G$ be a Lie group with a left-invariant SR-structure. Then the left-translation

$$L_g : G \to G : h \mapsto L_g h = g * h$$

obviously defines a SR isometry.

The Sub-Laplacian on nilpotent Lie groups

Carnot group

A Carnot group is a connected, simply connected Lie group G, with Lie algebra $\mathfrak{g}$ allowing a stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r.$$

Moreover, the following bracket relations hold:

$$[V_1, V_j] = V_{j+1}, \quad j = 1, \cdots, r-1,$$
$$[V_j, V_r] = \{0\}, \quad j = 1, \cdots, r.$$

In particular $\mathfrak{g}$ is nilpotent of step r.

Example: *Let $\mathfrak{h}_3$ be the Heisenberg Lie algebra. Then*

$$\mathfrak{h}_3 = \text{span}\{X, Y\} \oplus \text{span}\{Z\},$$

where $[X, Y] = Z$. This is a 2-step case.

Carnot group

Reminder (Lie's fundamental Theorem):

Corollary

For every finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ there is a connected, simply connected Lie group G which a Lie algebra isomorphic to $\mathfrak{g}$. Moreover, G is unique up to isomorphisms.

This leads to the notion of Carnot group.

Definition

Let $\mathfrak{g}$ be a Carnot Lie algebra. The connected, simply connected Lie group G (up to isomorphisms) with Lie algebra $\mathfrak{g}$ is called Carnot group.

Remark: If $\mathfrak{g}$ has step r, we call the Carnot group G of step r.

Example: Engel group

Consider the Engel group $\mathcal{E}_4 \cong \mathbb{R}^4$ as a matrix group:

$$\mathcal{E}_4 = \left\{ \begin{pmatrix} 1 & x & \frac{x^2}{2} & z \\ 0 & 1 & x & w \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \; : \; x, y, w, z \in \mathbb{R} \right\} \subset \mathbb{R}^{4\times 4}.$$

Then $\mathcal{E}_4$ has the Lie algebra $\mathfrak{e}_4$ with non-trivial bracket relations:

$$[X, Y] = W \qquad \text{und} \qquad [X, \underbrace{[X, Y]}_{=W}] = Z$$

and stratification

$$\mathfrak{e}_4 = \operatorname{span}\{X, Y\} \oplus \operatorname{span}\{W\} \oplus \operatorname{span}\{Z\}.$$

Corollary

The Engel group $\mathcal{E}_4$ is a Carnot group of step 3.

Nilpotent approximation

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a regular SR manifold. Let $q \in M$ and recall:

$$\begin{aligned} \operatorname{gr}(\mathcal{H})_q &= \mathcal{H}_q \oplus \mathcal{H}_q^2/\mathcal{H}_q \oplus \cdots \oplus \mathcal{H}_q^r/\mathcal{H}_q^{r-1} \\ &= \text{nilpotentization}. \end{aligned}$$

Observations:

- Already discussed: Lie brackets of vector fields on M induce a Lie algebra structure on $\operatorname{gr}(\mathcal{H})_q$. *(respecting the grading).*
- Let $\operatorname{Gr}(\mathcal{H})_q$ denote the connected, simply connected nilpotent Lie group with Lie algebra $\operatorname{gr}(\mathcal{H})_q$.
- The space $\mathcal{H}_q \subset \operatorname{gr}(\mathcal{H})_q$ induces for each $q \in M$ a (left-invariant) SR structure on the group $\operatorname{Gr}(\mathcal{H})_q$ (Example of talk 1).

Definition

The group $\operatorname{Gr}(\mathcal{H})_q$ with the induced SR structure is called nilpotent approximation of the SR manifold M at $q \in M$.

Nilpotent approximation

Conclusion:

Carnot groups seem to be a local model of the SR manifold. It may be helpful to first study the Sub-Laplacian and subelliptic heat flow there.

Question

What is the intrinsic Sub-Laplacian on a Carnot group or (more generally) on any nilpotent Lie group?

Exponential coordinates: Let $(G, *)$ be a connected, simply connected nilpotent Lie group of dimension $\dim G = n$ and with Lie algebra $\mathfrak{g}$. Then

$$\exp : \mathfrak{g} \to G$$

is a diffeomorphism. Hence we can pullback the product on G to $\mathfrak{g} \cong \mathbb{R}^n$ via exp *(exponential coordinates)*.

Exponential coordinates

We have an identification:

$$(G, *) \cong (\mathfrak{g} \cong \mathbb{R}^n, \circ),$$

where

$$g \circ h := \log\Big(\exp(g) * \exp(h)\Big), \qquad \text{for all} \qquad g, h \in \mathfrak{g}.$$

Baker-Campbell-Hausdorff formula

Let $g, h \in \mathfrak{g}$, then

$$\exp(g) * \exp(h) =$$
$$= \exp\left(g + h + \frac{1}{2}[g,h] + \frac{1}{12}[g,[g,h]] - \frac{1}{12}[h,[g,h]] \mp \cdots\right)$$

Note: if $\mathfrak{g}$ is nilpotent, then the sum in the exponent is always finite.

Exponential coordinates

Using this formula above gives:

$$g \circ h = g + h + \frac{1}{2}[g,h] + \frac{1}{12}\big[g,[g,h]\big] - \frac{1}{12}\big[h,[g,h]\big] \mp \cdots \text{(finite)}.$$

Example

Consider the case $r = \text{step}\,\mathfrak{g} = 2$ and choose a decomposition

$$\mathfrak{g} = V_1 \oplus V_2$$

such that

$$[V_1, V_1] = V_2 \quad \textit{and} \quad [V_1, V_2] = [V_2, V_2] = 0.$$

Consider the SR structure on $\mathfrak{g} \cong G$ defined by:

$$\mathcal{H} = V_1 = \text{"left-invariant vector fields."}$$

Sub-Laplacian on nilpotent Lie groups

Example (continued)

Consider an inner product $\langle \cdot, \cdot \rangle$ on V_1 and chose an orthonormal basis:

$$[X_1, \cdots, X_m] = \text{"orthonormal basis of } V_1\text{"}.$$

Chose a basis $[Y_{m+1}, \cdots, Y_n]$ of V_2. Then there are structure constants c_{ij}^k such that

$$[X_i, X_j] = \sum_{\ell=m+1}^{n} c_{ij}^{\ell} Y_\ell, \qquad [X_i, Y_\ell] = 0 = [Y_\ell, Y_h].$$

This choice of basis gives a concrete identification $\mathfrak{g} \cong \mathbb{R}^n$.

Goal: *Calculate the left-invariant vector fields corresponding to the basis elements X_i explicitly in coordinates of $\mathbb{R}^n$.*

Sub-Laplacian on nilpotent Lie groups

Example (continued)

Let $f \in C^\infty(\mathbb{R}^n)$ and $g = \sum_{j=1}^m x_j X_j \in \mathfrak{g}$. Then

$$\begin{aligned}
[X_i f](g) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f\Big(g \circ tX_i\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f\Big(g + tX_i + \frac{1}{2}[g, tX_i]\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f\Big(g + tX_i + \frac{t}{2}\sum_{j=1}^m x_j [X_j, X_i]\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f\Big(g + tX_i + \frac{t}{2}\sum_{j=1}^m \sum_{\ell=m+1}^n x_j c_{ji}^\ell Y_\ell\Big) \\
&= \Big\{\frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{j=1}^m \sum_{\ell=m+1}^n x_j c_{ij}^\ell \frac{\partial}{\partial y_\ell}\Big\} f(g).
\end{aligned}$$

Sub-Laplacian on nilpotent Lie groups

Example (continued)

We can identify $X_i \in V_1 \subset \mathfrak{g}$ with the following left-invariant vector field on $G \cong \mathbb{R}^n$:

$$\widetilde{X}_i := \frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{j=1}^m \sum_{\ell=m+1}^n x_j c_{ij}^\ell \frac{\partial}{\partial y_\ell}.$$

Observations:

- the coefficients in front of $\frac{\partial}{\partial x_i}$ is one for $i = 1, \cdots, m$,
- in the double sum the variable x_i does not appear ($c_{ii}^\ell = 0$ for all ℓ).

Let $\mathcal{P}$ = Lebesgue measure be the Popp measure on $G \cong \mathbb{R}^n$.

Goal: *Calculate the $\mathcal{P}$-divergence of X_i for $i = 1, \cdots, m$:*

From the above observations:

$$\mathcal{L}_{\widetilde{X}_i}\Big(dx_1 \wedge \cdots \wedge dx_m \wedge dy_{m+1} \wedge \cdots \wedge dy_{n-m}\Big) = d \circ \iota_{\widetilde{X}_i}\mathcal{P} = d\Big(\mathcal{P}(\widetilde{X}_i, \cdot)\Big) = 0.$$

Therefore $\mathrm{div}_{\mathcal{P}}(X_i) = 0$ for all $i = 1, \cdots, m$.

Sub-Laplacian on nilpotent Lie groups

Example (continued)

Conclusion: In the above example of a step-2 nilpotent Lie group we have found:

$$\Delta_{\text{sub}} = \sum_{i=1}^{m} \Big[\widetilde{X}_i^2 + \underbrace{\text{div}_\omega(X_i)}_{=0} X_i \Big] = \sum_{i=1}^{m} \widetilde{X}_i^2.$$

Hence, the intrinsic sub-Laplacian has no first order terms. We say:

$$\Delta_{\text{sub}} = \textit{sum-of-squares-operator.}$$

Hypoellipticity

Theorem (L. Hörmander, 1967)

Let $\Omega \subset \mathbb{R}^n$ be open. Consider C^∞- vector fields $[X_0, \cdots, X_m]$ with

$$\text{rank Lie}\Big[X_0, \cdots, X_m\Big] = n, \quad x \in \Omega \qquad \textit{(Hörmander condition).}$$

The differential operator $\mathcal{L}$ is hypoelliptic:

$$\mathcal{L} := \sum_{j=1}^{m} X_j^2 + X_0 + c \qquad c \in C^\infty(\Omega).$$

Remarks

- An operator P is called hypoelliptic if
$$Pu = f \quad \text{with} \quad f, u \in \mathcal{D}'(\Omega)$$
implies: Let $\Omega_0 \overset{\text{open}}{\subset} \Omega$ and $f \in C^\infty(\Omega_0)$, then $u \in C^\infty(\Omega_0)$.
- The hypoellipticity statement in the Hörmander's Theorem follows from subelliptic estimates:
$$\|u\|_{s-\delta} \leq C_D(\|Au\|_s + \|u\|_0), \qquad u \in C_0^\infty(\underset{\substack{\uparrow \\ \text{bounded domain}}}{D})$$
- In particular, elliptic operators (e.g. the Laplace operator on a Riemannian manifold) are hypoelliptic (elliptic regularity).

Hörmander theorem: the version on manifolds

Theorem (L. Hörmander, 1967)

Let $\mathcal{L}$ be a differential operator on a manifold M, that locally in a neighborhood U of any point is written as
$$\mathcal{L} = \sum_{i=1}^m X_i^2 + X_0,$$
where $X_0, X_1, \cdots, X_m$ are C^∞ - vector fields with
$$\text{Lie}_q\Big\{X_0, X_1, \cdots, X_m\Big\} = T_qM \quad \forall\ q \in U.$$
Then $\mathcal{L}$ is hypoelliptic. In particular:

The intrinsic sub-Laplacian on a SR-manifold M is hypoelliptic.

Example: Kolmogorov operator

At the beginning of the 20th century:

A prototype of a kind of operator studied by A. N. Kolmogorov in relation with diffusion phenomena is the following:

Example: *Kolmogorov operator* (proto-type)

$$K=\sum_{j=1}^{n}\partial_{x_j}^2+\sum_{j=1}^{n}x_j\partial_{y_j}-\partial_t, \qquad mit \quad (x,y,t)\in\mathbb{R}^{2n+1}$$

"sum of squares + a first order term.

$x=$ *velocity* and $y:=$*position*.

Operator with *non-negative degenerate characteristic form*.

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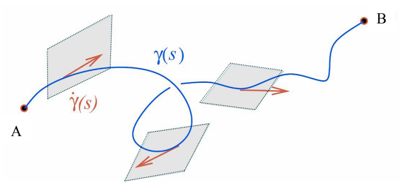
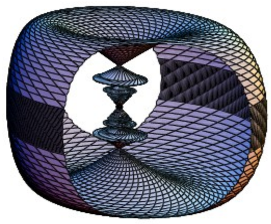
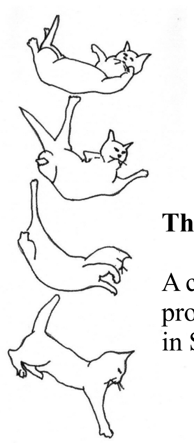
Thank you for your attention!



Distribution and horizontal curve



Front of SR geodesics at time T
(picture by: U. Boscain, D. Barilari)



The falling cat:

A connectivity problem in SR geometry

Appedix: The Hausdorff volume and Popp volume

Question: *How to choose the smooth measure ω in the ω-divergence?*

Requirement

If we would like to have a "geometric operator" such measure should only depend on the internal data of the SR-structure.

Possible candidates:

- The Hausdorff measure of (M, d_{cc})? (next slides)
- The Popp measure on $\mathcal{P}$ (next slides). This measure is a-priori smooth by construction.

Remark

- Maybe both measures coincide?
- If we have a "canonical measure" ω we may consider the sub-Riemannian heat equation:

$$\partial_t - \Delta_{\text{sub}} = 0$$

  and study its geometric significance in comparison with the Riemannian setting.

Hausdorff measure

Let (M, d) be a metric space and $\Omega \subset M$. Let

- $\varepsilon, s > 0$,
- $\{U_\alpha\}_\alpha$ a covering of Ω by open sets.

Consider:

$$\mu_\varepsilon^s(\Omega) := \inf\Big\{ \sum_\alpha \big[\text{diam } U_\alpha\big]^s \; : \; \forall \alpha : \; \text{diam } U_\alpha < \varepsilon \Big\}.$$

Hausdorff measure

The value

$$\mu^s(\Omega) := \lim_{\varepsilon \to 0} \mu_\varepsilon^s(\Omega) \in [0, \infty) \cup \{\infty\}$$

is called s-dimensional Hausdorff measure of Ω.

Proposition: *There is a unique value Q, the Hausdorff dimension of Ω, with $\mu^s(\Omega) = \infty$ for $s < Q$ and $\mu^s(\Omega) = 0$ for $s > Q$.*

Hausdorff measure

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Sub-Riemannian manifold. Then (M, d_{cc}) is a metric space with:

$$d_{cc}(A, B) = \inf\Big\{ \underset{\substack{\uparrow \\ \text{SR- length of } \gamma}}{L_{\text{SR}}(\gamma)} \; : \; \gamma(0) = A \quad \gamma(1) = B, \quad \gamma \text{ horizontal} \Big\}$$

$$= \textit{Carnot-Carathéodory distance on } M.$$

Definition

Let μ_{Haus}^Q be the Hausdorff measure of the metric space (M, d_{cc}).

Problem:

- it is hard to calculate μ_{Haus}^Q in general.
- not clear whether (or in which cases) the Hausdorff measure is a smooth measure on M.