Heat kernel of the Sub-Laplacian on the Heisenberg group

3. lecture

"Singular Integrals on nilpotent Lie groups and related topics"

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Outline

1. Subriemannian heat kernel: revisited
2. The Grushin operator: a simple model
3. Complex Hamilton Jacobi method
4. Subelliptic heat kernel: From Grushin operator to sub-Laplacian and back.

The SR heat kernel: revisited

Aim: *We look for explicit formulas for the heat kernel of the Sub-Laplace operator on nilpotent Lie groups.*

Definition

The heat kernel of the sub-Laplacian Δ_{sub} on an SR manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ denoted by:

$$K(t; x, y) : (0, \infty) \times M \times M \longrightarrow \mathbb{R}$$

is the fundamental solution of the heat operator:

$$P := \frac{\partial}{\partial t} - \Delta_{\text{sub}},$$

i.e. $K(t; x, y)$ fulfills

$$\begin{cases} PK(t; \cdot, y) = 0, & \textit{for all } t > 0 \\ \lim_{t \downarrow 0} K(t; x, \cdot) = \delta_x, & \textit{in the distributional sense.} \end{cases}$$

The SR heat kernel: revisited

From now on assume: (M, d_{CC}) is complete as a metric space.

Remarks: Abstractly, the following is known:

- Δ_{sub} is essentially selfadjoint on $C_c^\infty(M)$. Existence and uniqueness of the heat kernel is guaranteed. [1]
- Hörmander's Theorem also implies the hypoellipticity of the SR heat operator
  $$P := \frac{\partial}{\partial t} - \Delta_{\text{sub}}.$$
  In fact, the SR heat kernel K solves
  $$PK(t; \cdot, y) = 0.$$
- K is symmetric in the space variables, i.e. $K(t; x, y) = K(t; y, x)$. Hence, K is a smooth kernel.

[1] Robert S. Strichartz. Sub-Riemannian geometry. J. Differential Geom., 24(2):221 - 263, 1986.

The heat kernel: A bridge between analysis and geometry

Intuition: Let $x, y \in M$, *(Riemannian manifold)*:

heat kernel $= K(t; x, y) =$ "*heat flowing from* x *to* y *at time* t."

"Meta-Theorem", (Not a precise mathematical statement)

The heat kernel of the sub-Laplacian Δ_{sub} has the form of a path integral:

$$K(t; x, y) = \int_{P_t(x,y)} e^{-S_t(\gamma)} d\mu_t(\gamma).$$

(i) $P_t(x, y) =$ space of horizontal curves, connecting x and y.

(ii) $S_t(\gamma)$ is a classical action $S_t(\gamma) = \frac{1}{2} \int_0^1 \|\dot{\gamma}(s)\|^2 ds$.

(iii) μ_t, a "measure" on the infinite dimensional space $P_t(x, y)$.

The heat kernel: A bridge between analysis and geometry

Question: *How to calculate heat kernels in some examples (if possible)?*

In general, it is not possibly to calculate the heat kernel explicitly (and may not even be useful). However,

Remark

- For specific classes of subelliptic operators (including some sub-Laplace operators) methods and formulas are available, e.g. [a]

  "Complex Hamilton-Jacobi method".

- Asymptotic properties are more easily obtained even without having explicit formulas.

[a] O. Calin, D.-C. Chang, K. Furutani, C. Iwasaki, *Heat kernels for elliptic and sub-elliptic operators. Methods and techniques*, Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2011.

A model operator

Aim: *To get an idea we start with a model operator for which we can calculate the heat kernel "by hand".*

Consider the 3-dimensional Heisenberg group $(\mathbb{H}_3 \cong \mathbb{R}^3, *)$ with product

$$(x_1, y_1, z_1) \; * \; (x_2, y_2, z_2) = \Big(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\Big).$$

Corresponding Heisenberg Lie algebra:

$$\mathfrak{h}_3 = \text{span}\Big\{X, Y, Z\Big\} \quad \textit{where} \quad [X, Y] = Z,$$

where X, Y, Z are left-invariant vector fields on $\mathbb{H}_3$:

(Hypoelliptic) sub-Laplace operator

$$\Delta_{\text{sub}} = \frac{1}{2}\Big(X^2 + Y^2\Big) = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}\right)^2.$$

A model operator

Model: *For simplicity we reduce the space dimension from three to two.*

Consider the (abelian) subgroup:

$$N_Y = \Big\{(0, t, 0) \; : \; t \in \mathbb{R}\Big\} \subset (\mathbb{H}_3, *)$$

and the projection $\pi : \mathbb{H}_3 \longrightarrow N_Y \backslash \mathbb{H}_3 : g \mapsto N_Y g$ onto the left-quotient.

Lemma

The map ρ below is well-defined and invertible (a diffeomorphism):

$$\rho : N_Y \backslash \mathbb{H}_3 \to \mathbb{R}^2 : N_Y * (x, y, z) \mapsto \big(x, z + \frac{xy}{2}\big) \in \mathbb{R}^2.$$

Well-definedness: Let $t \in \mathbb{R}$:

$$N_Y * (0, t, 0) * (x, y, z) = N_Y * \left(x, y + t, z - \frac{xt}{2}\right) \overset{\rho}{\mapsto}$$
$$\left(x, z - \frac{xt}{2} + \frac{1}{2}x(y + t)\right) = \left(x, z + \frac{xy}{2}\right).$$

A model operator

Via composition we obtain a map:

$$\tilde{\pi} = \rho \circ \pi : \mathbb{H}_3 \xrightarrow{\pi} N_Y \backslash \mathbb{H}_3 \underset{\cong}{\xrightarrow{\rho}} \mathbb{R}^2 : (x, y, z) \mapsto (x, z + \frac{xy}{2}).$$

As usual let $(\tilde{\pi})^*$ denotes the pullback of functions along $\tilde{\pi}$:

$$(\tilde{\pi})^* : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{H}_3) : f \mapsto f \circ \tilde{\pi}.$$

Lemma

There is a second order differential operator $\mathcal{G}$ on $\mathbb{R}^2$ (called Grushin operator) such that:

$$\Delta_{\text{sub}} \circ (\tilde{\pi})^* = (\tilde{\pi})^* \circ \mathcal{G}. \tag{1}$$

With coordinates (u, v) of $\mathbb{R}^2$ it has the simple form:

$$\mathcal{G} = \frac{1}{2}\left(\frac{\partial^2}{\partial u^2} + u^2 \frac{\partial^2}{\partial v^2}\right) = \text{"sum-of-squares."}$$

A model operator

Remark

Note that the vector fields $V = \frac{\partial}{\partial u}$ and $W = u\frac{\partial}{\partial v}$ are linearly dependent exactly on the line

$$\mathcal{S} := \{(u, v) = (0, v) \ : \ v \in \mathbb{R}\} \subset \mathbb{R}^2.$$

The Grushin operator

$$\mathcal{G} = \frac{1}{2}(V^2 + W^2)$$

is the Laplace operator on $\mathbb{R}^2 \setminus \mathcal{S}$ (Grushin plane) with respect to a Riemannian metric, which becomes singular at $\mathcal{S}$.

Next plan:

We study the (subordinate to Δ_{sub} on $\mathbb{H}_3$) Grushin operator $\mathcal{G}$ on $\mathbb{R}^2$ and calculate its heat kernel via an explicit spectral decomposition.

Spectral decomposition and Mehler formula:

First step: Perform a partial Fourier transform in the operator $\mathcal{G}$.

$$\mathcal{F}^y : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) : \big[\mathcal{F}^y f\big](x,\eta) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x,y) e^{-iy\eta} dy.$$

Using the rule $\frac{\partial}{\partial y}\mathcal{F}^y = -i\mathcal{F}^y \eta$ we obtain the differential operator:

$$L_\eta := \left(\mathcal{F}^y\right)^{-1} \circ \mathcal{G} \circ \mathcal{F}^y = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\eta^2\right), \qquad \eta \in \mathbb{R}.$$

Note: L_η is closely related to the well-understood Hermite operator.

Idea

We interpret $(L_\eta)_{\eta\in\mathbb{R}}$ as a parameter family of operators on $\mathbb{R}$. Now, perform a spectral decomposition of each operator L_η

From spectral decomposition to the heat kernel

Let A be an operator on $L^2(\mathbb{R})$ and $[\varphi_j : j \in \mathbb{N}] \subset \mathcal{S}(\mathbb{R})$ an orthonormal basis consisting of eigenfunctions with eigenvalues

$$0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_j \ldots \longrightarrow -\infty \quad \text{"fast" as } j \to \infty.$$

Ansatz: Then the heat kernel of A should have the form:

$$K(t;g,h) = \sum_{j=1}^{\infty} e^{t\lambda_j} \varphi_j(g)\overline{\varphi_j(h)} \tag{2}$$

(in case of convergence). In fact, let $f \in \mathcal{S}(\mathbb{R})$:

$$\Big(\frac{\partial}{\partial t} - A\Big)K(t;\cdot,h) = \sum_{j=1}^{\infty} e^{t\lambda_j} \underbrace{\Big(\lambda_j\varphi_j - A\varphi_j\Big)}_{=0} \overline{\varphi_j(h)} = 0.$$

$$\begin{aligned}\lim_{t\downarrow 0} \int_{\mathbb{R}^2} f(h)K(t;g,h)dh &= \lim_{t\downarrow 0} \sum_{j=1}^{\infty} e^{t\lambda_j} \int_{\mathbb{R}^2} f(h)\varphi_j(g)\overline{\varphi_j(h)}dh \\ &= \sum \langle f, \varphi_j\rangle_{L^2}\varphi_j(g) = f(g) = \delta_g(f).\end{aligned}$$

From spectral decomposition to the heat kernel

Observation: *We known the spectral decomposition of $A = L_\eta$ explicitly.*

Lemma (spectral decomposition of L_η)

Let $\eta \neq 0$ be fixed. Consider the n-th Hermite polynomial ($n \in \mathbb{N}_0$)

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n} \quad \text{and put} \quad V_n(x) := e^{-\frac{1}{2}|\eta|x^2} H_n(\sqrt{|\eta|}x).$$

Then V_n is an eigenfunction of L_η with:

- *eigenvalue $\lambda_n = -(n+\frac{1}{2})|\eta|$ of multiplicity one:*
- $\|V_n\|_{L^2}^2 = \sqrt{\frac{\pi}{|\eta|}} 2^n n!$,
- *and $[\frac{V_n}{\|V_n\|_{L^2}} : n \in \mathbb{N}_0]$ forms an orthonormal basis of $L^2(\mathbb{R})$.*

Mehler formula

The previous observation provides the heat kernel of L_η for $\eta \neq 0$:

$$\begin{aligned} K^\eta(t;x,\tilde{x}) &= \sum_{n=0}^\infty e^{-(n+\frac{1}{2})|\eta|t} \frac{\sqrt{|\eta|} V_n(x) V_n(\tilde{x})}{\sqrt{\pi} 2^n n!} \\ &= \sqrt{|\eta|} e^{-\frac{1}{2}|\eta|t} e^{-\frac{|\eta|}{2}(x^2+\tilde{x}^2)} \sum_{n=0}^\infty \frac{H_n(\sqrt{|\eta|}x) H_n(\sqrt{|\eta|}\tilde{x})}{\sqrt{\pi} 2^n n!} e^{-nt|\eta|}. \end{aligned}$$

In order to calculate the infinite sum we use the Mehler formula:

Lemma (*Mehler formula*)

Let $|w| < 1$, then:

$$\sum_{n=0}^\infty \frac{H_n(x) H_n(\tilde{x})}{2^n n!} w^n = \sqrt{\frac{1}{1-w^2}} e^{-\frac{(x+\tilde{x})^2}{4} \frac{w-1}{w+1} - \frac{(x-\tilde{x})^2}{4} \frac{w+1}{w-1}}.$$

Heat kernel of the Grushin operator

Lemma

The heat kernel of the operator $L_\eta = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\eta^2\right)$ *with* $\eta \neq 0$ *has the form:*

$$K_\eta(t;x,\tilde{x}) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{\eta}{e^{t\eta}-e^{-t\eta}}}\,e^{-\frac{\eta}{4}\left\{(x+\tilde{x})^2\tanh\frac{\eta t}{2}+(x-\tilde{x})^2\coth\frac{\eta t}{2}\right\}}.$$

Note that:

$$\sqrt{\frac{\eta}{e^{t\eta}-e^{-t\eta}}} = \frac{1}{\sqrt{2}}\sqrt{\frac{\eta}{\sinh t\eta}}$$

is an even function in the variable η.

Remark: As $\eta \to 0$ we recover the well-known heat kernel of the Laplace operator L_0 on $\mathbb{R}$ which has no eigenvalues:

$$\lim_{\eta\to 0} K_\eta(t;x,\tilde{x}) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{\|x-\tilde{x}\|^2}{2t}}, \qquad (x,\tilde{x}\in\mathbb{R}).$$

Heat kernel of the Grushin operator

From the heat kernels of $(L_\eta)_{\eta\in\mathbb{R}}$ we calculate the heat kernel of $\mathcal{G}$:

Lemma

The heat kernel $K^{\mathcal{G}}$ *of the Grushin operator*

$$\mathcal{G} = \frac{1}{2}\left(\frac{\partial^2}{\partial u^2} + u^2\frac{\partial^2}{\partial v^2}\right) = \mathcal{F}^y \circ L_\eta \circ \left(\mathcal{F}^y\right)^{-1}$$

is obtained by applying the (inverse) Fourier transform to the family of heat kernels of L_η*:*

$$K^{\mathcal{G}}(t;x,y,\tilde{x},\tilde{y}) = \\ = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}} e^{i(y-\tilde{y})\eta}e^{-\frac{\eta}{4}\left\{(x+\tilde{x})^2\tanh\frac{t\eta}{2}+(x-\tilde{x})^2\coth\frac{t\eta}{2}\right\}}\sqrt{\frac{\eta}{\sinh t\eta}}\,d\eta.$$

Proof: Check that $K^{\mathcal{G}}$ has the properties of the heat kernel. Then, use uniqueness of the heat kernel (which also needs a proof).

How to generalize this?

The Grushin operator looked rather easy.

Ingredients to our proof

- An explicit spectral decomposition of the operators L_η when $\eta \neq 0$,
- Mehler formula, which gives an expression of the generating function for the Hermite functions.

Problem: *We were very lucky! However, for more general operators such tools may not be available.*

Question: *The heat kernel does not give a spectral decomposition.*

- Can we calculate the heat kernel of $\mathcal{G}$ without knowing the spectral decomposition explicitly and which we may be able to generalize?
- Can some geometry be helpful?

Idea: *Compare the heat kernel $K^{\mathcal{G}}$ with the form in the Meta Theorem.*

Heat kernel of Grushin operator: a second method

Rewrite the heat kernel of $\mathcal{G}$ by applying a time scaling, i.e. change $t\eta$ to η in the integral:

$$K^{\mathcal{G}}(t; x, y, \tilde{x}, \tilde{y}) = \\ = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}} e^{i\frac{(y-\tilde{y})\eta}{t}} e^{-\frac{\eta}{4t}\left\{(x+\tilde{x})^2 \tanh \frac{\eta}{2} + (x-\tilde{x})^2 \coth \frac{\eta}{2}\right\}} \sqrt{\frac{\eta}{\sinh \eta}} d\eta.$$

We rename the functions appearing in the integration as follows:

$$S(x, \tilde{x}, \eta) := \frac{\eta}{4}\left\{(x+\tilde{x})^2 \tanh \frac{\eta}{2} + (x-\tilde{x})^2 \coth \frac{\eta}{2}\right\},$$

$$V(\eta) := \sqrt{\frac{\eta}{\sinh \eta}} = \textit{volume element.}$$

Observation: The heat kernel $K^{\mathcal{G}}$ can be written in the form:

$$K^{\mathcal{G}}(t; x, y, \tilde{x}, \tilde{y}) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}} e^{i\frac{(y-\tilde{y})\eta}{t} - \frac{S(x,\tilde{x},\eta)}{t}} V(\eta) d\eta.$$

Heat kernel of Grushin operator: a second method

Observation: The heat kernel $K^{\mathcal{G}}$ can be written in the form:

$$K^{\mathcal{G}}(t;x,y,\tilde{x},\tilde{y}) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}} e^{i\frac{(y-\tilde{y})\eta}{t} - \frac{S(x,\tilde{x},\eta)}{t}} V(\eta) d\eta.$$

"Meta-Theorem"

The heat kernel has the form of a path integral:

$$K(t;x,y) = \int_{P_t(x,y)} e^{-S_t(\gamma)} d\mu_t(\gamma).$$

(i) $P_t(x,y) =$ space of horizontal curves, connecting x and y.

(ii) $S_t(\gamma) = \frac{1}{2}\int_0^1 \|\dot{\gamma}(s)\|^2 ds$ is a classical action

(iii) μ_t, a "measure" on the infinite dimensional space $P_t(x,y)$.

Heat kernel of Grushin operator: a second method

Aim: *Obtain $S = S(x,\tilde{x},\eta)$ from the solution of a Hamilton system under initial and end condition associated to the Grushin operator.*

Let $\eta \neq 0$ and consider the Hamiltonian H^η corresponding to the operator

$$L_\eta = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\eta^2\right).$$

Explicitly,

$$H^\eta(x,\xi) = \frac{1}{2}(\xi^2 - x^2\eta^2).$$

Let $x, \tilde{x} \in \mathbb{R}$ and $t > 0$. The induced Hamilton system is given by:

$$(HS): \quad \begin{cases} \dot{x}(s) &= \frac{\partial H^\eta}{\partial \xi} = \xi(s) \\ \dot{\xi}(s) &= -\frac{\partial H^\eta}{\partial x} = x(s)\eta^2 \\ x(0) &= x \quad \textit{and} \quad x(t) = \tilde{x} \qquad \textit{(initial and end condition).} \end{cases}$$

Heat kernel of Grushin operator: a second method

This system can be uniquely solved with explicit formulas:

$$x(s) = x(s;t,x,\tilde{x},\eta) = \frac{\tilde{x}\sinh(s\eta) + x\sinh\eta(t-s)}{\sinh(t\eta)}\}$$

$$\xi(s) = \xi(s;t,x,\tilde{x},\eta) = \dot{x}(s) = \eta\frac{\tilde{x}\cosh(s\eta) - x\cosh(t-s)\eta}{\sinh t\eta}.$$

From this solution we build the so-called *classical action*:

$$\varphi(x,\tilde{x},t;\eta) = \int_0^t \underbrace{(\dot{x}(s))^2 - H^\eta(x(s),\xi(s))}_{=L^\eta(t;x,\dot{x})} ds$$

$$= \text{"classical action"}.$$

Recall from ODE:

The integrand $L^\eta(t;x,\dot{x})$ is called Lagrange function. It is obtained by a Legendre transform of the Hamiltonian: $L^\eta = (H^\eta)^*$.

Heat kernel of Grushin operator: a second method

Remark

The Hamiltonian H^η is constant along solutions to the Hamilton system.

$$H^\eta(x(s),\dot{x}(s)) \equiv H^\eta(x(0),\xi(0)) \qquad (\dot{x} = \xi)$$

$$= \frac{1}{2}(\xi^2(0) - x^2\eta^2) := E = \text{"energy"}.$$

From the above expression of ξ:

$$\xi(0) = \eta\frac{\tilde{x} - x\cosh(t\eta)}{\sinh t\eta}.$$

Inserting these data into the integrand of φ gives (after a calculation):

$$\varphi(x,\tilde{x},t;\eta) = \int_0^t \underbrace{\dot{x}(s)^2 - tE}_{=L^\eta(t;x,\dot{x})} dt = \frac{\eta}{4}\Big\{(\tilde{x}+x)^2\tanh\frac{t\eta}{2} + (x+\tilde{x})^2\coth\frac{t\eta}{2}\Big\}.$$

Heat kernel of Grushin operator: a second method

Conclusion: The classical action φ in fact appears in the heat kernel expression of the Grushin operator $\mathcal{G}$.

Lemma

The function $S(x, \tilde{x}; \eta)$ appearing in the exponent of the heat kernel:

$$K^{\mathcal{G}}(t; x, y, \tilde{x}, \tilde{y}) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}} e^{i\frac{(y-\tilde{y})\eta}{t} - \frac{S(x,\tilde{x},\eta)}{t}} V(\eta) d\eta.$$

coincides with the classical action of the Hamiltonian system (HS) at the time $t = 1$:

$$S(x, \tilde{x}, \eta) = \varphi(x, \tilde{x}, 1; \eta) = \text{"classical action"} .$$

Hence: In order to find this part of the heat kernel we **need not** to "pass through" the spectrum of $\mathcal{G}$.

Heat kernel of Grushin operator: more equations

Know from PDE: the classical action φ solves the

"Hamilton-Jacobi equation".

Roughly speaking: the "PDE for the geodesic distance", i.e.:

$$\frac{\partial \varphi}{\partial t}(x, \tilde{x}, t; \eta) + H^{\eta}\Big(\tilde{x}, \frac{\partial}{\partial \tilde{x}}\varphi(x, \tilde{x}, t; \eta)\Big) = 0. \qquad \text{(HJE)}$$

Corollary (Generalized Hamilton Jacobi equation)

The function $S(x, \tilde{x}; \eta)$ in the exponent of the integrant of $K^{\mathcal{G}}$ solves the so-called "generalized Hamilton-Jacobi equation":

$$H^{\eta}\left(\tilde{x}, \frac{\partial}{\partial \tilde{x}} S(x, \tilde{x}; \eta)\right) + \eta \frac{\partial S}{\partial \eta}(x, \tilde{x}; \eta) = S(x, \tilde{x}; \eta).$$

Proof: (HJE) and $S(x, \tilde{x}; t\eta) = \varphi(x, \tilde{x}, 1; t\eta) = t\varphi(x, \tilde{x}, t; \eta)$.

Heat kernel of Grushin operator: more equations

Take a second look at the heat kernel of the Grushin operator:

$$K^{\mathcal{G}}(t;x,y,\tilde{x},\tilde{y}) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}} e^{i\frac{(y-\tilde{y})\eta}{t} - \frac{S(x,\tilde{x},\eta)}{t}} V(\eta) d\eta.$$

Question: *How to interpret the function $V(\eta)$ (the "volume element")?*

Correspondence

Fix the following values:

$$t = time, \quad x = initial\ condition \quad and \quad \eta \neq 0.$$

and consider the correspondence $\mathcal{V}$ between final condition $\tilde{x}$ and the value of the dual variable ξ at time $s = 0$:

$$\mathcal{V}(\cdot;t,x,\eta) : \tilde{x} \mapsto \xi(0;t,x,\tilde{x},\eta).$$

Since we have an explicit formula for ξ we obtain $\mathcal{V}$ explicitly, namely:

$$\mathcal{V}(\tilde{x};t,x,\eta) = \frac{\eta}{\sinh(t\eta)} (\tilde{x} - x\cosh(t\eta)).$$

Heat kernel of Grushin operator: more equations

We make the following observation:

Lemma

Let $V(\eta)$ be the volume element in the heat kernel expression.

(a) *The function V and $\mathcal{V}$ are related by the equation:*

$$\sqrt{\frac{\partial \mathcal{V}}{\partial \tilde{x}}(\tilde{x};t,x,\eta)} = \sqrt{\frac{\eta}{\sinh \eta}} = V(\eta).$$

(b) *The volume element solves the transport equation:*

$$\eta \frac{\partial V}{\partial \eta} - \Big(-\mathcal{G}S(0,\tilde{x};\eta) + \frac{1}{2}\Big) V = 0.$$

Question: *Can these observations for the low dimensional model of the Grushin operator be generalized to obtain the heat kernel of the sub-Laplacian Δ_{sub} on nilpotent Lie groups (without spectral decompositions)?*

Sub-Laplacian on step-2 nilpotent Lie groups

Let $(G, *)$ be a step-2 nilpotent Lie group with Lie algebra:

$$\mathfrak{g} = V_1 \oplus V_2,$$

such that

$$[V_1, V_1] = V_2 \quad \textit{and} \quad [V_1, V_2] = [V_2, V_2] = 0.$$

Consider an inner product $\langle \cdot, \cdot \rangle$ on V_1 and choose

$$\underbrace{[X_1, \cdots, X_m]}_{\text{left-invariant vector fields on } G} = \text{"orthonormal basis of } V_1\text{"}.$$

Choose now a basis $[Y_{m+1}, \cdots, Y_n]$ of V_2 and write:

$$[X_i, X_j] = \sum_{\ell=m+1}^{n} c_{ij}^{\ell} Y_\ell, \quad \textit{and} \quad [X_i, Y_\ell] = 0 = [Y_\ell, Y_h].$$

Definition

We call the skew-symmetric matrices $(c_{ij})_{ij}^{\ell} \in \mathbb{R}^{m \times m}$ for $\ell = m+1, \ldots, n$ the structure constants.

Sub-Laplacian on step-2 nilpotent Lie groups

Identity X_i with the left-invariant vector fields on $\mathbb{R}^n \cong G$:

$$\widetilde{X}_i := \frac{\partial}{\partial x_i} - \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=m+1}^{n} x_j c_{ij}^{\ell} \frac{\partial}{\partial y_\ell}.$$

Consider the left-invariant sub-Laplacian:

$$\Delta_{\text{sub}} = \frac{1}{2} \sum_{i=1}^{m} \widetilde{X}_i^2 = \frac{1}{2} \sum_{i=1}^{m} \Big[\frac{\partial}{\partial x_i} - \frac{1}{2} \sum_{j=1}^{m} \sum_{\ell=m+1}^{n} x_j c_{ij}^{\ell} \frac{\partial}{\partial y_\ell} \Big]^2.$$

Lemma

The heat kernel $K_{\text{sub}} \in C^\infty(\mathbb{R}_+ \times G \times G)$ is a "convolution kernel", i.e.

$$K_{\text{sub}}(t; g, h) = k(t, g^{-1} * h) \quad \textit{where} \quad k(t, g) \in C^\infty(\mathbb{R} \times G),$$

such that

(a) $\big(\frac{\partial}{\partial t} - \Delta_{\text{sub}}\big) k(t, g) = 0.$

(b) $\lim_{t \downarrow 0} k(t, \cdot) = \delta_e =$ *delta-distribution at $e \in G$, where e is the unit.*

Sub-Laplacian on step-2 nilpotent Lie groups

Question: *How can we find $k(t,g)$?*

According to the form of the heat kernel $K^{\mathcal{G}}$ for the Grushin operator (or based on the Meta theorem) we try the following Ansatz:

$$k(t,g) = \frac{1}{t^\rho} \int_{\mathbb{R}^d} e^{\frac{f(g,\eta)}{t}} V(g,\eta) d\eta. \qquad (3)$$

Here we have the following ingredients (which need to be determined):

- $\rho \geq 0$,
- $d = n - m = \dim V_2 =$ dimension of the center of $\mathfrak{g}$
- $f = f(g,\eta) \in C^\infty(G \times \mathbb{R}^d) =$ "complex action function".
- $V = V(g,\eta) \in C^\infty(G \times \mathbb{R}^d) =$ "volume element".

Idea: *Find conditions on ρ, f and V such that properties of the last Lemma hold.*

Complex Hamilton-Jacobi Theory

The corresponding analysis is called

"Complex Hamilton Jacobi theory".

Repeating what we did for the Gruhsin operator $\mathcal{G}$, it goes like this:

Method to determine f and V:

(a) Construct the complex action function $f(g,\eta)$ by uniquely solving a Hamiltonian system under

"initial-final conditions."

(b) Construct the volume element $V(\eta)$ from the Jacobian of the correspondence between the final and initial condition of the Hamiltonian system (van Vleck determinant).

Let $z = (z_1, \dots, z_\ell)^t \in \mathbb{R}^{n-m}$ and define the matrix-valued function:

$$\Omega(z) = \sum_{k=1}^{d} z_k \left(c_{ij}^k\right)_{i,j} \in \mathbb{R}^{m \times m}.$$

Heat kernel: the formula

Theorem (Beals-Gaveau-Greiner formula)

The integral kernel K_{sub} *(= heat kernel) of the heat operator*

$$\frac{\partial}{\partial t} - \Delta_{\text{sub}} \quad \textit{on} \quad \mathbb{R}_+ \times G$$

has the form:

$$K_{\text{sub}}(t,g,h) = k(t, g^{-1} * h) = \frac{1}{(2\pi t)^{m/2+d}} \int_{\mathbb{R}^d} e^{-\frac{f(g^{-1}*h,\eta)}{t}} V(\eta)\, d\eta,$$

Put $g = (x,z) \in \mathbb{R}^m \times \mathbb{R}^d$, *then:*

$$f(g,\eta) = f(x,z,\eta) = i\langle \eta, z\rangle + \frac{1}{2}\Big\langle \Omega(i\eta)\coth\big(\Omega(i\eta)\big)\cdot x, x\Big\rangle,$$

$$V(\eta) = \left\{\det \frac{\Omega(i\eta)}{\sinh \Omega(i\eta)}\right\}^{1/2}.$$

Heat kernel: related PDE

Let $H(x,\xi)$ denote the Hamiltonian of Δ_{sub}:

Remark

Generalizing our observation in the case of the Grushin operator the functions f and V solve certain PDE:

- The action function f solves the generalized Hamilton-Jacobi equation.

$$H(x,\nabla_g f) + \sum_{i=1}^{d} \eta_\ell \frac{\partial}{\partial \eta_\ell} f(g,\eta) = f(g,\eta). \tag{GHJE}$$

- With a solution $f(g,\eta)$ to Equation (GHJE) the volume element $V(g,\eta)$ solves the transport equation:

$$\sum_{i=1}^{\ell} \eta_i \frac{\partial V}{\partial \eta_i} - \Big(\Delta_{\text{sub}}(f) + \frac{m}{2}\Big) V = 0.$$

Heat kernel: sub-Laplacian on the Heisenberg group

Example: *We specialize the last theorem to the heat kernel of the sub-Laplacian on the Heisenberg group* $\mathbb{H}_3$.

Bracket relation: (*Heisenberg Lie algebra* $\mathfrak{h}_3$): Here $m = 2$ and $d = 1$:

$$[X, Y] = Z, \quad \textit{where} \quad \mathfrak{h}_3 = \operatorname{span}\{X, Y, Z\}.$$

We obtain the matrix of structure constants

$$\Omega(z) = z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}, \quad \textit{where} \quad z \in \mathbb{R}.$$

Observation: the matrices $\Omega(i\eta)$ are selfadjoint:

$$\Omega(i\eta) = \Omega(i\eta)^* \quad \textit{for all} \quad \eta \in \mathbb{R}$$

and can be diagonalized with eigenvalues

$$\lambda_\pm = \pm\eta.$$

Heat kernel: sub-Laplacian on the Heisenberg group

Here are all functions that appear in the representation of the heat kernel:

Ingredients to the heat kernel

- volume element: $V(\eta)$ is given by:

$$V(\eta)^2 = \det \begin{pmatrix} \frac{\eta}{\sinh \eta} & 0 \\ 0 & \frac{-\eta}{\sinh(-\eta)} \end{pmatrix} = \frac{\eta^2}{\sinh^2(\eta)}.$$

- action function: $f = f(x, y, z; \eta)$ is given by:

$$f(x, y, z; \eta) = i\eta z + \frac{\eta}{2}\coth(\eta)(x^2 + y^2).$$

- convolution: Let $g = (x, y, z), h = (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{H}_3$. Then,

$$g^{-1} * h = -g * h = \Big(-x + \tilde{x}, -y + \tilde{y}, -z + \tilde{z} + \frac{1}{2}(-x\tilde{y} + \tilde{x}y) \Big).$$

Heat kernel: sub-Laplacian on the Heisenberg group

Theorem

The heat kernel of the sub-Laplace operator Δ_{sub} on $\mathbb{H}_3$ has the explicit form:

$$K_{\text{sub}}(t; g, h) = k(t, g^{-1} * h)$$
$$= \frac{1}{(2\pi t)^2} \int_{\mathbb{R}} e^{i\eta\left(z-\tilde{z}+\frac{x\tilde{y}-\tilde{x}y}{2}\right)+\frac{\eta}{2t}\coth\eta\left\{\left(x-\tilde{x}\right)^2+(y-\tilde{y})^2\right\}} \cdot \frac{\eta}{\sinh\eta} d\eta.$$

From sub-Laplacian to Grushin

Grushin operator: revisited:

Recall: the Grushin operator $\mathcal{G}$ on $\mathbb{R}^2$:

$$\mathcal{G} = \frac{1}{2}\left(\frac{\partial^2}{\partial u^2} + u^2 \frac{\partial^2}{\partial v^2}\right)$$

is related to the sub-Laplacian Δ_{sub} on $\mathbb{H}_3$ via:

$$\Delta_{\text{sub}} \circ (\tilde{\pi})^* = (\tilde{\pi})^* \circ \mathcal{G},$$

where π is the canonical projection:

$$\pi : \mathbb{H}_3 \to N_Y \backslash \mathbb{H}_3 \cong \mathbb{R}^2 \text{ and } N_Y := \left\{(0, t, 0) \in \mathbb{H}_3 \,:\, t \in \mathbb{R}\right\} \overset{\text{subgroup}}{\subset} \mathbb{H}_3.$$

Aim: *From the above explicit expression of the heat kernel of Δ_{sub} we can re-obtain the heat kernel of $\mathcal{G}$ via a "fiber integration".*

From sub-Laplacian to Grushin

Here is the way it works:

Consider again the global trivialization of $\pi : \mathbb{H}_3 \to N_Y \backslash \mathbb{H}_3 \cong \mathbb{R}^2$:

$$\varphi : N_Y \times \big(N_Y \backslash \mathbb{H}_3\big) \cong \mathbb{R} \times \mathbb{R}^2 \ni \big(a, u, v\big) \mapsto \big(u, a, v - \frac{au}{2}\big) \in \mathbb{R}^3 \cong \mathbb{H}_3.$$

In particular, φ is a diffeomorphism with

$$\pi \circ \varphi\big(a, N_Y g\big) = N_Y g.$$

Lemma

The heat kernels $K^{\mathcal{G}}$ of $\mathcal{G}$ and K_{sub} of Δ_{sub} are related via:

$$K^{\mathcal{G}}\big(t; \underbrace{\pi(x)}_{\in \mathbb{R}^2}, y\big) = \int_{\mathbb{R}} K_{\text{sub}}\big(t; x, \varphi(\underset{\uparrow \text{"fiber variable"}}{a}, y)\big)\, da.$$

Here $x \in \mathbb{H}_3$ and $y \in \mathbb{R}^2$. Note that $\pi : \mathbb{H}_3 \to \mathbb{R}^2$ is surjective.

A Question

Question: *Can we generalize the Beals-Gaveau-Greiner Theorem and as well calculate the heat kernel of the sub-Laplacian on*

"Carnot groups of step $r > 2$"?

Maybe no: *As we have discussed in the last lecture in relation with the Engel group.*

Heat kernel: sub-Laplacian on the Heisenberg group

Theorem

The heat kernel of the sub-Laplace operator Δ_{sub} on $\mathbb{H}_3$ has the explicit form:

$$K_{\text{sub}}(t; g, h) = k(t, g^{-1} * h) = \frac{1}{(2\pi t)^2} \int_{\mathbb{R}} e^{i\eta\left(z-\tilde{z}+\frac{x\tilde{y}-\tilde{x}y}{2}\right)+\frac{\eta}{2t}\coth\eta\left\{\left(x-\tilde{x}\right)^2+(y-\tilde{y})^2\right\}} \cdot \frac{\eta}{\sinh\eta} d\eta.$$

Question: *Can we generalize the formula and calculate the heat kernel of the sub-Laplacian on*

"Carnot groups of step $r > 2$"?

What it is good for?

Subriemannian geodesics on the Heisenberg group $\mathbb{H}_3$

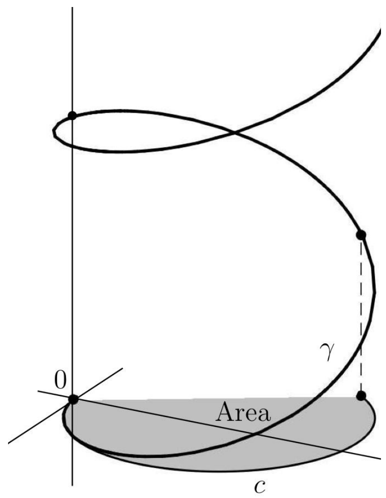


Figure: SR geodesic on $\mathbb{H}_3$ and isoperimetric problem in the plane.

Heat kernel/trace expansion

Even if we do not an explicit formula we may apply asymptotic results: Here are examples:

Theorem (Ben Arous, Leandré)

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a SR manifold and $q \in M$. Let $N \in \mathbb{N}$:

$$K(t, q, q) = \frac{1}{t^{\frac{Q(q)}{2}}} \Big(c_0(q) + c_1(q)t + \cdots c_N(q)t^N + O(t^{N+1}) \Big)$$

as $t \downarrow 0$. Here:

$$Q(q) = \textit{Hausdorff dimension with respect to the } d_{cc}\textit{-metric.}$$

Definition: We call the coefficients heat invariants.

Problem: *What is the "geometric content" of the heat invariants in this subelliptic setting, or $\cdots$*

"Can one hear the subriemannian structure?"

More asymptotic relations

Theorem (Leandré)

Let $x, y \in (M, \mathcal{H}, \langle \cdot, \cdot \rangle)$, then

$$\lim_{t \downarrow 0} t \log K(t, x, y) = -\frac{d_{cc}(x, y)^2}{2}.$$

The heat kernel contains information on the d_{cc}-metric.

Theorem

Let M be compact and equiregular. Then we have a heat trace expansion:

$$\operatorname{trace}\big(e^{t\Delta_{sub}}\big) \sim \frac{1}{t^{\frac{Q}{2}}} \Big(\alpha_0 + \alpha_1 t + \alpha_2 t^2 \cdots \Big) \qquad t \downarrow 0.$$

A Theorem, a Question and a first Answer:

G =*nilpotent Lie group* (e.g. nilpotentization) with lattice $\Gamma \subset G$.

$$M = \Gamma \backslash G = \textit{compact nilmanifold.}$$

Theorem (W. Bauer, K. Furutani, C. Iwasaki 2012)

Assume that G[a] is of step 2 and let $\Delta_{\text{sub}}^{\Gamma}$ be the intrinsic sub-Laplace operator on M. Then:

$$\text{trace}\left(e^{t\Delta_{\text{sub}}^{\Gamma}}\right) = \frac{C}{t^{\frac{m}{2}+d}} + O(t^{\infty}) \quad \text{as} \quad t \to 0.$$

Here C is explicitly known and encodes the Popp volume of M.

$$\dim M = m + d, \quad d = \dim \text{center } \mathfrak{g} \quad \longleftarrow \text{Lie algebra of G}$$

$$\frac{m}{2} + d = \frac{1}{2} \times \{\text{Hausdorff dimension of } (M, d_{cc})\}.$$

[a] e.g. G can be the Heisenberg group

Question:

Under the conditions of the last theorem:

Questions

(a) Which geometric data can we recover from the spectrum of the sub-Laplace operator (inverse spectral problem), e.g.:

Can we read from the spectrum of $\Delta_{\text{sub}}^{\Gamma}$ the manifold dimension $\dim M = m + d$?

(b) Does the theorem hold for nilpotent Lie groups of step ≥ 3?

Answer to (a): In some specific cases Yes, (K. Furutani, 2020). But unknown in general.

Answer to (b): Yes!

The short-time asymptotic expansion of the heat kernel on any nilmanifold contains only a single non-trivial term. This is true in an even more general setting (*V. Fischer, 2022*). [2]

[2] V. Fischer, *Asymptotic and zeta function on compact nilmanifolds*, J. Math. Pures Appl. 160, 1-28, 2022.

Conclusion:

Some intuition: The last result - roughly speaking - indicates:

Carnot groups (nilpotent Lie groups), which are the local models of a SR manifold are "flat" spaces in SR geometry.

However: they are **not flat** as Riemannian manifolds.

Next Aim

Consider certain "curved SR manifolds". Study the short time heat kernel asymptotic via the local models (step 2 Carnot groups).

Questions:

- What means curvature in this framework?
- Can we express the second heat invariant via curvature terms?

H-type foliation and second heat invariant

Aim: *We consider the intrinsic sub-Laplace operator on Clifford bundles in SR geometry. The local models are H-type groups.*

Short review on H-type foliations:

Let (M, g) be a Riemannian manifold with metric g and of dimension $\dim M = n + m$. Assume that M is equipped with a

"Riemannian foliation"

locally being a Riemannian submersion (with bundle-like metric).

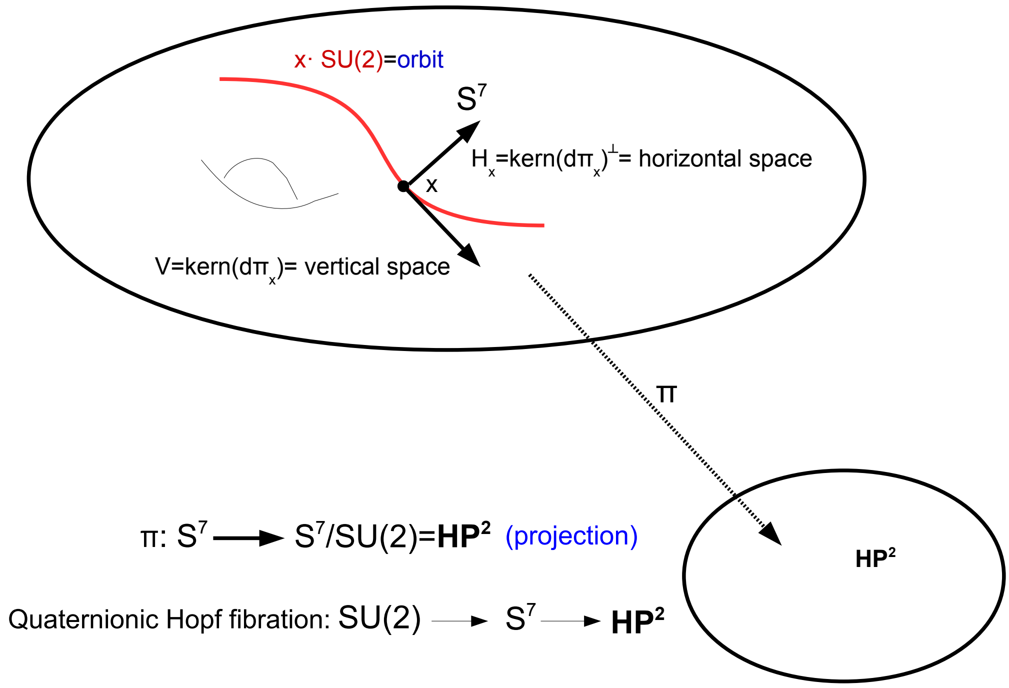
Example: Riemannian foliation may be induced by a Riemannian submersion (e.g. a principal bundle).

Define (locally)

$\mathcal{V}$ = *vertical bundle: formed by vectors tangent to the leaves,*

$\mathcal{H}$ = *horizontal bundle: orthogonal to* $\mathcal{V}$.

Example: Quaternionic Hopf fibration



H-type foliation and second heat invariant

Induced splitting of tangent spaces and metric: For all $q \in M$:

$$T_qM = \mathcal{H}_q \oplus \mathcal{V}_q \quad \text{and} \quad g = \underset{\substack{\uparrow \\ \text{restriction of } g \text{ to } \mathcal{H}}}{g_{\mathcal{H}}} \oplus g_{\mathcal{V}}.$$

Assumptions:

- bundle-like complete metric: for all $X \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$:

$$(\mathcal{L}_X g)(Z,Z) = 0.$$

  Geodesics tangent to $\mathcal{H}$ at some point remain tangent to $\mathcal{H}$.
- totally geodesic: for all $X \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$:

$$(\mathcal{L}_Z g)(X,X) = 0.$$

  All leaves are totally geodesic submanifolds.

H-type foliation and second heat invariant

Known: Under these assumptions there is a canonical connection ∇ on M preserving metric and foliation structure called

"Bott connection."

Theorem and Definition

The Bott connection on a totally-geodesic foliation with bundle-like metric is uniquely characterized by the following properties:

- (metric): $\nabla g = 0$,
- (compatible): For $X \in \Gamma(TM)$: $\nabla_X \mathcal{H} \subset \mathcal{H}$ and $\nabla_X \mathcal{V} \subset \mathcal{V}$,
- (torsion): The torsion

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

satisfies:

$$T(\mathcal{H},\mathcal{H}) \subset \mathcal{V} \quad \text{and} \quad T(\mathcal{H},\mathcal{V}) = T(\mathcal{V},\mathcal{V}) = 0.$$

H-type foliation and second heat invariant

Definition (J-map)

For every $Z \in \Gamma(\mathcal{V})$ define a bundle endomorphism $J_Z : \mathcal{H} \to \mathcal{H}$ by

$$g\big(J_Z X, Y\big) = g\big(Z, T(X,Y)\big).$$

The next result implies that $(M, \mathcal{H}, g_{\mathcal{H}})$ under a suitable condition defines a SR manifold:

Lemma

Suppose that the H-type condition:

$$J_Z^2 = -g(Z,Z)\mathrm{Id}_{\mathcal{H}} \quad \textit{for all} \quad Z \in \Gamma(\mathcal{V})$$

is satisfied. Then T_qM at any $q \in M$ is generated by $[X,\mathcal{H}]_q$ and $\mathcal{H}_q$ for every horizontal vector field $X \in \Gamma(\mathcal{H})$ with $X_q \neq 0$.

Remark: We call $\mathcal{H}$ strongly bracket generating or fat.

H-type foliation and second heat invariant

Definition (H-type foliation)

The SR manifold $(M, \mathcal{H}, g_{\mathcal{H}})$ is called an H-type foliation if the H-type condition is satisfied.

Remark:

- This class contains many classical examples.
- Recently such foliations were studied (also under additional assumptions) in:

F. Baudoin, E. Grong, L. Rizzi, G. Vega-Molino, *H-type foliations*, arXiv 2021.

Some examples of (compact) H-type foliations

Structure	Torsion
Complex Type, $m = 1, n = 2k$	
K-Contact	YM
Sasakian	CP
Heisenberg Group	CP
Hopf Fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2k+1} \to \mathbb{C}P^k$	CP
Anti de-Sitter Fibration $\mathbb{S}^1 \hookrightarrow \mathbf{AdS}^{2k+1}(\mathbb{C}) \to \mathbb{C}H^k$	CP
Twistor Type, $m = 2, n = 4k$	
Twistor space over quaternionic Kähler manifold	HP
Projective Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}P^{2k+1} \to \mathbb{H}P^k$	HP
Hyperbolic Twistor space $\mathbb{CP}^1 \hookrightarrow \mathbb{C}H^{2k+1} \to \mathbb{H}H^k$	HP
Quaternionic Type, $m = 3, n = 4k$	
3K-contact	YM
Negative 3K-contact	YM
3-Sasakian	HP
Negative 3-Sasakian	HP
Torus bundle over hyperkähler manifolds	CP
Quaternionic Heisenberg Group	CP
Quaternionic Hopf Fibration $\mathbf{SU}(2) \hookrightarrow \mathbb{S}^{4k+3} \to \mathbb{H}P^k$	HP
Quaternionic Anti de-Sitter Fibration $\mathbf{SU}(2) \hookrightarrow \mathbf{AdS}^{4k+3}(\mathbb{H}) \to \mathbb{H}H^k$	HP
Octonionic Type, $m = 7, n = 8$	
Octonionic Heisenberg Group	CP
Octonionic Hopf Fibration $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{O}P^1$	HP
Octonionic Anti de-Sitter Fibration $\mathbb{S}^7 \hookrightarrow \mathbf{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1$	HP
H-type Groups, m is arbitrary	CP

H-type foliation and second heat invariant

Consider now a local horizontal frame $X_1, \dots, X_n$ of $\mathcal{H}$, i.e. X_j are pointwise orthonormal horizontal vector fields such that

$$\mathcal{H}_q = \text{span}\Big\{X_1, \dots, X_n\Big\}_q \qquad \textit{for all} \qquad q \in M.$$

Correspondingly, consider the metric dual frame $\{\theta^1, \dots, \theta^n\}$.

curvature

The Bott connection induces a curvature tensor in the usual way:

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

H-type foliation and second heat invariant

Definition (horizontal scalar curvature)

We define the following "*horizontal objects*":

$$R^{\delta}_{\alpha\beta\gamma} := \theta^{\delta}\Big(R(X_\alpha, X_\beta)X_\gamma\Big) \qquad \textit{with} \qquad \alpha, \beta, \gamma, \delta = 1, \dots, n.$$

The horizontal scalar curvature of the Bott connection is given by:

$$\kappa_{\mathcal{H}} := \sum_{\alpha,\beta=1}^{n} R^{\alpha}_{\alpha\beta\beta}.$$

Note: the value of $\kappa_{\mathcal{H}}$ is independent of the choice of the orthonormal horizontal frame and of its vertical complement.

H-type foliation and second heat invariant

We define a second local invariant which is a *"vertical object"*.

Definition

For vertical vector fields $Z, W \in \Gamma(\mathcal{V})$ consider the bundle-like operator

$$M(Z, W) : \Gamma(\mathcal{H}) \longrightarrow \Gamma(\mathcal{H})$$

defined by

$$M(Z, W)X := J_W J_Z (\nabla_Z J)_W X.$$

With a given orthonormal frame $\{Z_1, \ldots, Z_m\}$ of the vertical distribution $\mathcal{V}$ we define the function:

$$\tau_{\mathcal{V}} := \sum_{i,j=1}^{m} \overset{\substack{\text{matrix trace} \\ \downarrow}}{\text{trace}} \Big(M\left(Z_i, Z_j\right) \Big).$$

Note: $\tau_{\mathcal{V}}$ is independent of the choice of the vertical frame.

Example

Under some additional assumptions on the H-type foliation we can interpret the vertical quantity $\tau_{\mathcal{V}}$ more geometrically:

Theorem

Let $m \geq 2$. Assume that

- the torsion T is horizontally parallel, i.e.

$$\nabla_X T = 0, \qquad X \in \Gamma(\mathcal{H}).$$

- the sectional curvature $\kappa_{\mathcal{V}}$ of the leaves is a positive constant.

Then we have:

$$\tau_{\mathcal{V}} = m(m-1)\sigma\sqrt{\kappa_{\mathcal{V}}},$$

with $\sigma \in \mathbb{Z}$ being the difference between positive and negative eigenvalues of the symmetric part of $M(Z, W)$, where Z, W are any linear independent vertical vector fields.

H-type foliation and second heat invariant

Now we can formulate our main result:

Theorem (W.-B., I. Markina, A. Laaroussi, G. Vega-Molino, 2022)

Let $(M, \mathcal{H}, g_{\mathcal{H}})$ be an H-type foliation with intrinsic sub-Laplace operator

$$\Delta_{\text{sub}} = \text{div}_{\omega_{\text{Popp}}} \circ \text{grad}_{\mathcal{H}}.$$

Moreover, assume that the torsion induced by the Bott connection is horizontally parallel, i.e. $\nabla_{\mathcal{H}} T = 0$.

- *With $q \in M$ the heat kernel K_{sub} of Δ_{sub} has a short time asymptotic expansion of the form*

$$K_{\text{sub}}(t; q, q) = \frac{1}{t^{\frac{n}{2}+m}} \Big(c_0(q) + c_1(q)t + O(t^3) \Big) \quad \text{as} \quad t \downarrow 0.$$

H-type foliation and second heat invariant

Theorem (continued)

- *The second heat invariant $c_1(q)$ is a linear combination of the local invariants $\kappa_{\mathcal{H}}$ and $\tau_{\mathcal{V}}$ above:*

$$c_1(q) = C_1 \cdot \kappa_{\mathcal{H}}(q) + C_2 \cdot \tau_{\mathcal{V}}(q), \qquad q \in M,$$

where C_1 and C_2 are universal constants only depending on $n = \text{rank}\, \mathcal{H}$ and $m = \text{rank}\, \mathcal{V}$.

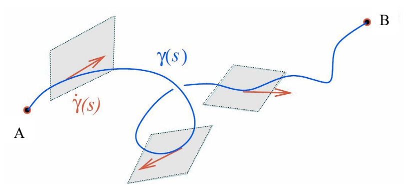
References

W.-B. I. Markina, A. Laaroussi, G. Vega-Molino
Local Invariants and Geometry of the sub-Laplacian on H-type Foliations, arXiv:2209.02168, 2022.

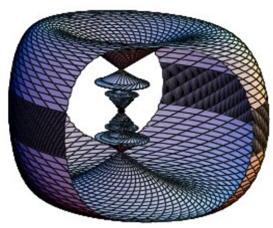
R. Beals, B. Gaveau, P. Greiner,
Hamilton-Jacobi theory and the heat kernel on Heisenberg groups, J. Math. Pures Appl. 79(7) (2000), 633-689.

O. Calin, D.-C. Chang, K. Furutani, C. Iwasaki,
Heat kernels for elliptic and sub-elliptic operators. Methods and techniques, Appl. Numer Harmonic Analysis. Birkhäuser/Springer, New York, 2011.

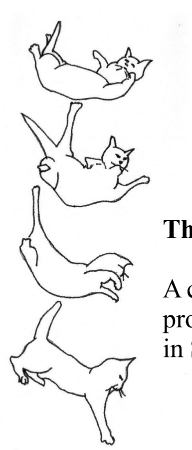
Thank you for your attention!



Distribution and horizontal curve



Front of SR geodesics at time T
(picture by: U. Boscain, D. Barilari)



The falling cat:

A connectivity problem in SR geometry