Global lifting and fundamental solution of degenerate operators

5. lecture

"Singular Integrals on nilpotent Lie groups and related topics"

Summer school, Universität Göttingen

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Sept. 19-23, 2022

Outline

1. Fundamental solutions and homogeneous vector fields
2. A global lifting theorem by Folland
3. Fundamental solutions via liftings

Fundamental solution

Consider a linear PDO [1] P on $\mathbb{R}^n$ of order d, i.e.

$$P = \sum_{|\alpha| \leq d} a_\alpha(x) D_x^\alpha \quad \textit{where} \quad a_\alpha(x) \in C^\infty(\mathbb{R}^n, \mathbb{R})$$

where we use the standard notation

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \textit{for} \quad \alpha \in \mathbb{N}_0^n.$$

Definition

Call $\Gamma : \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \, : \, x \neq y\} \to \mathbb{R}$ a global fundamental solution if

- For every $x \in \mathbb{R}^n$ we have $\Gamma(x, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^n)$.
- For every $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \Gamma(x, y) P^* \varphi(y) dy = -\varphi(x).$$

[1] PDO=partial differential operator

Fundamental solution

Remarks

- The defining equation of the fundamental solution is shortly written:

$$P\Gamma_x = -\delta_x = \textit{Dirac distribution supported at } x.$$

- Existence of the fundamental solution is not always guaranteed. Showing existence (or non-existence) can be complicated.
- In general, fundamental solutions are not unique, e.g. one may add a P-harmonic function h i.e.

$$Ph = 0$$

to a fundamental solution and gets another one.

From the heat kernel to the inverse

Let $\mathcal{L} = \mathcal{L}^*$ be a partial differential operator on $\mathbb{R}^n_x$ (smooth coefficients).

Definition

The corresponding heat operator on $\mathbb{R}^n_x \times \mathbb{R}_t$ is defined by $\mathcal{H} = \mathcal{L} - \partial_t$.

Assumption:

$\mathcal{H}$ admits a heat kernel $\{p_t(x,y)\}_{t>0}$ which is integrable with respect to t. In particular:

$$\mathcal{L}_y p_t(x,y) = \partial_t p_t(x,y).$$

Observation: At least formally, a fundamental solution of $\mathcal{L}$ is given by

$$\Gamma(x,y) = \int_0^\infty p_t(x,y)dt.$$

(This can be made precise in many cases).

From the heat kernel to the inverse

In fact (formal calculation): Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} \Gamma(x,y)\mathcal{L}^*\varphi(y)dy &= \int_{\mathbb{R}^n}\int_0^\infty p_t(x,y)dt\mathcal{L}^*\varphi(y)dy \\
&= \int_{\mathbb{R}^n}\int_0^\infty \mathcal{L}_y p_t(x,y)dt\varphi(y)dy \\
&= \int_{\mathbb{R}^n}\int_0^\infty \partial_t p_t(x,y)dt\varphi(y)dy \\
&= \int_{\mathbb{R}^n} \Big[p_t(x,y)\Big]_0^\infty \varphi(y)dy \\
&= -\lim_{t\to 0}\int_{\mathbb{R}^n} p_t(x,y)\varphi(y)dy = -\varphi(x).
\end{aligned}$$

Here we have assumed that $\lim_{t\to\infty} p_t(x,y) = 0$.

Reducing the dimension

Here is another example:

Consider the Laplace operator Δ_n on $\mathbb{R}^n$ with $n > 2$ and let $p \geq 1$.

$$\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \quad \text{and} \quad \Delta_{n+p} = \Delta_n + \sum_{j=n+1}^{n+p} \frac{\partial^2}{\partial x_j^2}.$$

Then Δ_n has the fundamental solution

$$p_n(x, y) := c_n \|x - y\|^{2-n}.$$

Observation

We can consider Δ_{n+p} for $p \geq 1$ as a

"lifting of Δ_n",

i.e. it acts on functions only depending on the variable $x_1, \cdots, x_n$ as Δ_n.

Reducing dimension

Lemma

We obtain the fundamental solution of Δ_n from the fundamental solution of Δ_{n+p} via a "fiber integration".

$$c\left(\sqrt{x_1^2 + \cdots + x_n^2}\right)^{2-n} =$$

$$= \int_{\mathbb{R}^p} \left(\sqrt{x_1^2 + \cdots + x_n^2 + t_1^2 + \cdots + t_p^2}\right)^{2-n-p} dt_1 \cdots dt_p = (*).$$

Proof: The change of variables $t = \|x\|\tau$ with $\tau \in \mathbb{R}^p$ and $x \neq 0$ gives:

$$(*) = \|x\|^p \int_{\mathbb{R}^p} \left(\|x\|^2 + \|\tau\|^2\|x\|^2\right)^{\frac{2-n-p}{2}} d\tau$$

$$= \|x\|^{2-n} \underbrace{\int_{\mathbb{R}^p} (1 + \|\tau\|^2)^{\frac{2-n-p}{2}} d\tau}_{=:c}.$$

□

Lifting

We describe recent work by **S. Biagi and A. Bonfiglioli**: [2]

Let P be a PDO with smooth coefficients.

Definition

We call a PDO P_{lift} on $\mathbb{R}^n \times \mathbb{R}^p$ a lifting of P if

(a) P_{lift} has smooth coefficients depending on $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p$.

(b) For every $f \in C^\infty(\mathbb{R})$:

$$P_{\text{lift}}(f \circ \pi)(x, \xi) = (Pf)(x) \quad \textit{where} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p.$$

Here $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the projection to the x-coordinates:

$$\pi(x, y) = x.$$

[2] S. Biagi, A. Bonfiglioli, *The existence of a global fundamental solution for homogeneous Hörmander operators via a global lifting method*, Proc. London Math. Soc. (3), 114 (2017), 855-889.

Example and equivalent formulation

Observation: P_{lift} is a lifting of P if and only if

$$P_{\text{lift}} = P + R \quad \textit{where} \quad R = \sum_{\beta \neq 0} r_{\alpha,\beta}(x, \xi) D_x^\alpha D_\xi^\beta.$$

Example

Consider the Grushin operator on $\mathbb{R}^2$:

$$\mathcal{G} := (\partial_{x_1})^2 + (x_1 \partial_{x_2})^2.$$

A lifting of $\mathcal{G}$ to $\mathbb{R}^3$ is given by:

$$\widetilde{\mathcal{G}} = (\partial_{x_1})^2 + (\partial_\xi + x_1 \partial_{x_2})^2 = \mathcal{G} + \underbrace{\partial_\xi^2 + 2x_1 \partial_{x_2} \partial_\xi}_{R}.$$

Remark: *$\widetilde{\mathcal{G}}$ is up to a change of variables the sub-Laplacian of the Heisenberg group. Its heat kernel is known do to the group structure.*

Questions

Assumption

Let P_{lift} be a lifting of P and P_{lift} admits a global fundamental solution Γ.

One may ask:

- *Does P admit a fundamental solution $\widetilde{\Gamma}$?*
- *If yes, can we obtain $\widetilde{\Gamma}$ by "via integration over some variables" in Γ?*
- *If we do not know Γ explicitly, but if we have estimates on Γ. Can we use them to obtain estimates on $\widetilde{\Gamma}$?*
- *Why should we lift at all? Adding more variables should make life more complicated.*

Homogeneous group and dilations

Let $G = (\mathbb{R}^n, *)$ be a nilpotent Lie group.

Definition (homogeneous group)

We call G a homogeneous group, it there is $\sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^n$ with

$$1 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$$

such that the dilation $\delta_\lambda : G \to G$ with

$$\delta_\lambda(x_1, \cdots, x_n) = \left(\lambda^{\sigma_1} x_1, \cdots, \lambda^{\sigma_n} x_n\right)$$

is an automorphism of G for every $\lambda > 0$, i.e.

$$\delta_\lambda(g) * \delta_\lambda(h) = \delta_\lambda\big(g * h\big).$$

Remark:

The dilations $\{\delta_\lambda\}_\lambda$ form a one-parameter group of automorphisms.

The homogeneous structure of the Heisenberg group

Example: Consider again the Heisenberg group $\mathbb{H}_3 \cong \mathbb{R}^3$ with dilation:

$$\delta_\lambda(x_1, x_2, x_3) := (\lambda x_1, \lambda x_2, \lambda^2 x_3) \quad \textit{with} \quad (\lambda > 0).$$

Then we have

$$\begin{aligned}
&\delta_\lambda(x_1, x_2, x_3) * \delta_\lambda(y_1, y_2, y_3) \\
&\quad = (\lambda x_1, \lambda x_2, \lambda^2 x_3) * (\lambda y_1, \lambda y_2, \lambda^2 y_3) \\
&\quad = \Big(\lambda(x_1 + y_1), \lambda(x_2 + y_2), \underbrace{\lambda^2(x_3 + y_3) + \frac{1}{2}\big[\lambda x_1 \lambda y_2 - \lambda y_1 \lambda x_2\big]}_{=\lambda^2\left((x_3+y_3)+\frac{1}{2}\left[x_1 y_2 - y_1 x_2\right]\right)}\Big) \\
&\quad = \delta_\lambda\Big((x_1, x_2, x_3) * (y_1, y_2, y_3)\Big).
\end{aligned}$$

Therefore $\mathbb{H}_3$ is a homogeneous Lie group with $\sigma = (1, 1, 2)$.

Homogeneous vector fields

Having dilations $\{\delta_\lambda\}_\lambda$ we can define δ_λ-homogeneous vector fields:

Definition

Let $X_1, \cdots, X_m$ be C^∞-vector fields in $G = (\mathbb{R}^n, *)$. Then we call X_j homogeneous of degree 1 if:

$$X_j\big(f \circ \delta_\lambda\big) = \lambda\big(X_j f\big) \circ \delta_\lambda \quad \forall\, \lambda > 0, \ \forall\, f \in C^\infty(\mathbb{R}^n, \mathbb{R}).$$

Further assumptions: Assume that $X_1, \cdots, X_m$

- are linearly independent as linear differential operators,
- fulfill the Hörmander bracket generating condition, i.e for all $g \in G$:

$$\dim \Big\{ X(g) \, : \, X \in \mathrm{Lie}\{X_1, \cdots, X_m\} \Big\} = n.$$

Note: *If X and Y are δ_λ-homogeneous of degree d_1 and d_2, respectively. Then $[X, Y]$ is δ_λ-homogeneous of degree $d_1 + d_2$.*

Example:

Consider again the Grushin operator $\mathcal{G}$ on $\mathbb{R}^2$ defined by

$$\mathcal{G} = (\partial_{x_1})^2 + (x_1\partial_{x_2})^2 = X_1^2 + X_2^2,$$

On $\mathbb{R}^2$ define the dilation $\delta_\lambda(x_1, x_2) := (\lambda x_1, \lambda^2 x_2)$.

Observation

- The bracket generating condition is fulfilled with $m = 2$, since

$$\dim\Big\{X_1 = \partial_{x_1}, X_2 = x_1\partial_{x_2}, [X_1, X_2] = \partial_{x_2}\Big\} = 3.$$

- X_1 and X_2 are homogeneous of degree 1: Let $g = (x_1, x_2)$, then

$$X_1(f \circ \delta_\lambda)(g) = \partial_{x_1}[f(\lambda x_1, \lambda^2 x_2)] = \lambda(\partial_{x_1} f)(\lambda x_1, \lambda^2 x_2) = \lambda(X_1 f) \circ \delta_\lambda(g),$$
$$X_2(f \circ \delta_\lambda)(g) = x_1\partial_{x_2}[f(\lambda x_1, \lambda^2 x_2)] = \lambda(\lambda x_1)[\partial_{x_2} f] \circ \delta_\lambda(g) = \lambda(X_2 f) \circ \delta_\lambda(g).$$

From homogeneous vector fields to a nilpotent Lie algebra

Let $\mathcal{X} = \{X_1, \cdots, X_m\}$ be homogeneous vector fields on $\mathbb{R}^n$ with the previous assumptions. We consider the Lie algebra generated by $\mathcal{X}$:

$$\begin{aligned}\mathfrak{a} :&= \mathrm{Lie}\Big\{X_1, \cdots, X_m\Big\}\\ &= \textit{smallest Lie subalgebra of vector fields on } \mathbb{R}^n \textit{ containing } \mathcal{X}.\end{aligned}$$

The δ_λ-homogeneity of the vector fields implies the following:

Lemma

The Lie algebra $\mathfrak{a}$ is finite dimensional and it corresponds to a Carnot group

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_r \quad \textit{and} \quad \begin{cases}[\mathfrak{a}_1, \mathfrak{a}_{i-1}] = \mathfrak{a}_i, & 2 \le i \le r,\\ [\mathfrak{a}_1, \mathfrak{a}_r] = \{0\}.\end{cases}$$

Here, the "first level" is $\mathfrak{a}_1 = \mathrm{span}\{X_1, \cdots, X_m\}$.

From the nilpotent Lie algebra to a Carnot group

Reminder:

We can equip $\mathfrak{a}$ with a group structure via exponential coordinates: Via the Campbell-Baker-Hausdorff formula the product is:

$$X \diamond Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \text{(finite)}.$$

Summing up:

Lemma

Let $N := \dim \mathfrak{a}$. Then $G = (\mathfrak{a} \cong \mathbb{R}^N, \diamond)$ is a Carnot group with Lie algebra (isomorphic to) $\mathfrak{a}$. Moreover,

$$\mathfrak{a} = \text{Lie}\Big\{X_1, \cdots, X_m\Big\}$$

is a Lie algebra of smooth vector fields on $\mathbb{R}^n$ (we can "exponentiate").

A family of dilations on $\mathfrak{a} \cong G$

Recall that $\mathfrak{a}$ has a decomposition:

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_r \quad \textit{with} \quad [\mathfrak{a}_1, \mathfrak{a}_{i-1}] = \mathfrak{a}_i.$$

Definition

For each $\lambda > 0$ define a dilation $\{\delta_\lambda^{\mathfrak{a}}\}_\lambda$ on $\mathfrak{a} \cong \mathbb{R}^N$ via the decomposition of elements:

$$\delta_\lambda^{\mathfrak{a}}(X) = \sum_{k=1}^r \lambda^k a_k \quad \textit{where} \quad X = \sum_{k=1}^r a_k \quad \textit{and} \quad a_k \in \mathfrak{a}_k.$$

Lemma: *The dilation $\delta_\lambda^{\mathfrak{a}}$ defines a group automorphism of $(G = \mathfrak{a}, \diamond)$.*

Proof: It is sufficient to show that $\delta_\lambda^{\mathfrak{a}}$ induces a Lie algebra automorphism:

$$[\delta_\lambda^{\mathfrak{a}}(X), \delta_\lambda^{\mathfrak{a}}(Y)] = \Big[\sum_{j=1}^r \lambda^j a_j, \sum_{\ell=1}^k \lambda^\ell b_\ell\Big] = \sum_{j,\ell=1}^r \lambda^{j+\ell} \underbrace{[a_j, a_\ell]}_{\in \mathfrak{a}_{j+\ell}} = \delta_\lambda^{\mathfrak{a}}([X,Y]).$$

$\square$.

Reminder: Subriemannian structure on a Carnot group

Definition

We call $(G \cong \mathfrak{a}, \diamond, \delta_\lambda^{\mathfrak{a}})$ a homogeneous Carnot group.

Next: Equip $(G \cong \mathfrak{a}, \diamond)$ with a Subriemannian structure:

Choose a linear basis of $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_r$ as follows:

- take the basis $[X_1, \cdots, X_m]$ of the first level $\mathfrak{a}_1$.
- for each $j = 2, \cdots, r$ take a basis $[X_1^{(j)}, \cdots X_{\ell_j}^{(j)}]$ of $\mathfrak{a}_j$.

Definition

We call the basis

$$\Big[X_1, \cdots, X_m, X_1^{(2)}, \cdots X_{\ell_2}^{(2)}, \cdots, X_1^{(r)}, \cdots, X_{\ell_r}^{(r)}\Big]$$

an adapted basis of the Lie algebra $\mathfrak{a}$. This basis gives the concrete identification between $\mathfrak{a}$ and $\mathbb{R}^N$.

Subriemannian structure on the Carnot group

Identifications

Via the above basis we make the following identifications:

$$\text{Carnot group: } G \cong \mathfrak{a} \quad \longleftrightarrow \quad \mathbb{R}^N,$$
$$\text{dilation on } G\text{: } \delta_\lambda^{\mathfrak{a}} \quad \longleftrightarrow \quad D_\lambda(a) = \big(\lambda^{s_1} a_1, \cdots, \lambda^{s_N} a_N\big), \quad a \in \mathbb{R}^N.$$

The exponents s_j in the dilation are given by:

$$(s_1, \cdots, s_N) = \big(\underbrace{1, \cdots, 1}_{= \dim \mathfrak{a}_1}, \underbrace{2, \cdots, 2}_{\dim \mathfrak{a}_2}, \cdots, \underbrace{r, \cdots, r}_{\dim \mathfrak{a}_r}\big).$$

Identify $[X_1, \cdots, X_m]$ with left-invariant vector fields $[J_1, \cdots, J_m]$ on the homogeneous Carnot group $(\mathbb{R}^N, \diamond, D_\lambda)$. Then

$$\mathcal{H} = \text{span}\Big\{J_1, \cdots, J_m\Big\} \subset T\mathbb{R}^N$$

is a bracket generating distribution in the tangent bundle of $\mathbb{R}^N$.

Sub-Laplacian on $\mathbb{R}^N$

Observation

The homogeneous Carnot group $(\mathbb{R}^N, \diamond, D_\lambda)$ is equipped via $\mathcal{H}$ with a Subriemannian structure. Its (intrinsic) Sub-Laplacian has the form:

$$\Delta_{\mathrm{sub},G} = J_1^2 + \cdots + J_m^2,$$

and defines a hypo-elliptic operator with underlying group structure.

Question: Now we have constructed two "sum-of-squares operators":

$$\begin{aligned} \mathcal{L} &= X_1^2 + \cdots + X_m^2 \quad (\text{on } \mathbb{R}^n) \\ \Delta_{\mathrm{sub},G} &= J_1^2 + \cdots + J_m^2, \quad (\text{on } G = \mathbb{R}^N \text{ where } N > n). \end{aligned}$$

- What is the relation between these operators?
- Can we use knowledge on $\Delta_{\mathrm{sub},G}$ to study $\mathcal{L}$?

From the Carnot group back to $\mathbb{R}^n$

Let $X \in \mathfrak{a} = \mathrm{Lie}\{X_1, \cdots, X_m\}$ be a δ_λ-homogeneous vector field on $\mathbb{R}^n$. Consider the induced integral curve starting in $0 \in \mathbb{R}^n$:

$$\Psi_t^X : \mathbb{R} \to \mathbb{R}^n \quad \text{with} \quad \begin{cases} \frac{d}{dt}\Psi_t^X = X \circ \Psi_t^X, & t \in \mathbb{R} \\ \Psi_0^X = 0. \end{cases} \qquad (*)$$

On the completeness

Based on the δ_λ-homogeneity one can show that all vector fields $X \in \mathfrak{a}$ are complete, i.e. the induced flow $(*)$ exists for all times $t \in \mathbb{R}$.

Consider the following map:

$$\pi : \mathbb{R}^N \to \mathbb{R}^n, \qquad \pi(a) = \left(\Psi_t^{X_a}(0)\right)_{|t=1},$$

where $\mathfrak{a} \ni X_a \longleftrightarrow a \in \mathbb{R}^N$ in our identification above.

Lifting theorem by Folland

Theorem (Folland, 1977)

The map $\pi : \mathbb{R}^N \to \mathbb{R}^n$ *has the following properties:*

- *for all* $\lambda > 0$ *and all* $a \in \mathbb{R}^N$ *we have:* [a]

$$\pi\Big(D_\lambda(a)\Big) = \delta_\lambda(\pi(a)).$$

- π *is a polynomial map.*
- *If* $J_1, \cdots, J_N$ *are the left-invariant vector fields which correspond to the adapted basis of* $\mathfrak{a} \cong \mathbb{R}^N$*, then*

$$d\pi(J_i)(a) = X_i\big(\pi(a)\big), \quad \forall\, a \in \mathbb{R}^N,$$

where X_i *is in the adapted basis of* $\mathfrak{a}$*.*

[a] G.B. Folland, *on the Rothschild-Stein lifting theorem*, Comm. Partial Differential Equations 2 (1977), 161-207.

Reminder: lifting of an operator

Definition

We call a PDO P_{lift} on $\mathbb{R}^n \times \mathbb{R}^p$ a lifting of P if

(a) P_{lift} has smooth coefficients depending on $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p$,

(b) For every $f \in C^\infty(\mathbb{R})$:

$$P_{\text{lift}}(f \circ \pi)(x, \xi) = (Pf)(x) \quad \textit{where} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p.$$

Here $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the projection to the x-coordinates:

$$\pi(x, y) = x.$$

Next:

One can choose coordinates in Folland's lifting theorem in such a way that

$$\pi : \mathbb{R}^N \to \mathbb{R}^n$$

becomes just the projection onto the first n coordinates of a vector in $\mathbb{R}^N$.

Lifting sums of squares

Theorem (S. Biagi, A. Bonfiglioli, 2017)

Let $X_1, \cdots, X_m$ be δ_λ-homogeneous of degree 1 vector fields on $\mathbb{R}^n$ with

$$N = \dim \mathrm{Lie}\{X_1, \cdots, X_m\}.$$

Then there is:

- *a homogeneous Carnot group $G = (\mathbb{R}^N, \diamond, D_\lambda)$ with m generators and nilpotent of step r.*
- *a system $\{Z_1, \cdots, Z_m\}$ of Lie generators of the Lie algebra $\mathfrak{a}$ of G such that*

  $$Z_i \text{ is a lifting of } X_i$$

  via the projection $\pi : \mathbb{R}^N \to \mathbb{R}^n$ onto the first n variables.

Remark: One can construct the lifting explicitly!

Lifting sums of squares

With the above notation we have:

Theorem (S. Biagi, A. Bonfiglioli, 2017)

The sub-Laplacian

$$\Delta_{\mathrm{sub},G} = Z_1^2 + \cdots + Z_m^2$$

on the homogeneous Carnot group $(\mathbb{R}^N, \diamond, D_\lambda)$, of the previous theorem is a lifting of the sum-of-squares operator:

$$\mathcal{L} = \sum_{k=1}^{m} X_k^2.$$

Fundamental solution

Assumptions:

Let $X_1, \cdots, X_m$ be linearly independent smooth vector fields on $\mathbb{R}^n$ with:

1. X_j for $j = 1, \cdots, m$ is δ_λ-homogeneous of degree 1.
2. Hörmander rank condition at zero:

$$\dim \Big\{ X(0) \, : \, X \in \text{Lie}\{X_1, \cdots, X_m\} \Big\} = n.$$

3. Define the sum-of-squares operator: $\mathcal{L} = \sum_{j=1}^m X_j^2$.
4. $G = (\mathbb{R}^N, \diamond, D_\lambda)$ = homogeneous Carnot group constructed above with sub-Laplacian:

$$\Delta_{\text{sub},G} = Z_1^2 + \cdots + Z_m^2.$$

A theorem by Folland

Homogeneous norm: $(\sigma_1, \cdots, \sigma_n, \sigma_1^*, \cdots, \sigma_p^*)$ hom. dimensions of D_λ:

$$h(x, \xi) = \sum_{j=1}^n |x_j|^{\frac{1}{\sigma_j}} + \sum_{k=1}^p |\xi_k|^{\frac{1}{\sigma_k^*}} \quad \textit{where} \quad (x, \xi) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p.$$

Theorem (G.B. Folland, 1973)

The sub-Laplacian $\Delta_{\text{sub},G}$ admits a unique fundamental solution γ_G with:

(a) $\gamma_G \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ *and* $\gamma_G > 0$ *on* $\mathbb{R}^N \setminus \{0\}$,

(b) $\gamma_G \in L^1_{\text{loc}}(\mathbb{R}^N)$ *and* γ_G *vanishes at infinity,*

(c) γ_G *is* D_λ*-homogeneous of degree* $2 - (\sum_{j=1}^n \sigma_j + \sum_{j=1}^p \sigma_j^*) = 2 - Q$ *and:*

$$\Delta_{\text{sub},G}(\gamma_G) = -\delta_0.$$

(d) *There is* $C > 0$ *with:* $C^{-1} h^{2-Q}(x, \xi) \le \gamma_G(x, \xi) \le C h^{2-Q}(x, \xi)$.

Fundamental solution of $\mathcal{L}$

$$\Gamma_G(x,\xi;y,\eta) = \gamma_G\Big((x,\xi)^{-1}\diamond(y,\eta)\Big), \qquad (x,\xi)\neq(y,\eta).$$

Theorem (S. Biagi, A. Bonfiglioli, 17)

Assume that $q=\sum_{j=1}^n \sigma_j > 2$ *and* $G=(\mathbb{R}^N,\diamond,D_\lambda)$ = *as above.*

(a) *Then*

$$\Gamma(x,y) := \int_{\mathbb{R}^p} \Gamma_G(x,0;y,\eta)\,d\eta \qquad (x\neq y)$$

is a fundamental solution for $\mathcal{L}=X_1^1+\cdots+X_m^2$.

(b) *There is a global estimate:*

$$C^{-1}\int_{\mathbb{R}^p} h^{2-Q}\big((x,0)^{-1}\diamond(y,\eta)\big)\,d\eta \le \Gamma(x,y) \le$$
$$\le C\int_{\mathbb{R}^p} h^{2-Q}\big((x,0)^{-1}\diamond(y,\eta)\big)\,d\eta.$$

Fundamental solution of $\mathcal{L}$ (continued)

Theorem (S. Biagi, A. Bonfiglioli, 17)

With the previous notations: $\Gamma(x,y)$ *has the* δ_λ*-homogeneity:*

Put : $\quad \Gamma\Big(\delta_\lambda(x);\delta_\lambda(y)\Big) = \lambda^{2-q}\cdot\Gamma(x,y)$ *where* $\mathbb{R}^n \ni x\neq y,\ \lambda>0.$

Moreover $\Gamma(x,y)$ *has the following properties:*

(1) *Symmetry:* $\Gamma(x,y)=\Gamma(y,x)$ *for all* $x\neq y\in\mathbb{R}^n$,

(2) $\Gamma(x,\cdot)=\Gamma(\cdot,x)$ *is* $\mathcal{L}$*-harmonic on* $\mathbb{R}^n\setminus\{x\}$,

(3) $\Gamma(x,\cdot)=\Gamma(\cdot,x)$ *vanishes at infinity uniformly on compact sets,*

(4) *Outside the diagonal* $\mathrm{Diag}=\{(x,x)\,:\,x\in\mathbb{R}^n\}$ *in* $\mathbb{R}^n\times\mathbb{R}^n$*:*

$$\Gamma\in L^1_{\mathrm{loc}}(\mathbb{R}^n\times\mathbb{R}^n\setminus\mathrm{Diag})\cap C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\mathrm{Diag}).$$

Example

Consider the Grushin operator on $\mathbb{R}^2$ with dilation $\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2)$:

$$\mathcal{L} = X_1^2 + X_2^2 \quad \textit{where} \quad X_1 = \partial_{x_1}, \quad X_2 = x_1 \partial_{x_2}.$$

- Carnot group: $G = (\mathbb{R}^3, \diamond, D_\lambda)$ with

$$D_\lambda(x_1, , x_2, \xi) = (\lambda x_1, \lambda^2 x_2, \lambda \xi) \quad \textit{and} \quad Q = 4.$$

- Product on G:

$$(x_1, x_2, \xi) \diamond (y_1, y_2, \eta) = \big(x_1 + y_1, x_2 + y_2 + x_1 \eta, \xi + \eta\big).$$

- Liftings of $X_1 \to Z_1 = \partial_{x_1}$ and $X_2 \to Z_2 = x_1 \partial_{x_2} + \partial_\xi$ and

$$\mathcal{L} = X_1^2 + X_2^2 \quad \textit{lifts to} \quad \Delta_{\text{sub}} = Z_1^2 + Z_2^2.$$

Example

- Fundamental solution of Δ_{sub} for $(x, \xi) \neq (0, 0)$:

$$\Gamma_{\Delta_{\text{sub}}}(x, \xi) = \frac{c}{\sqrt{(x_1^2 + \xi^2)^2 + 16(x_2 - \frac{1}{2}x_1\xi)^2}}.$$

Conclusion

The fundamental solution of $\mathcal{L}$ is given by the fiber integral:

$$\Gamma(x_1, x_2; y_1, y_2) =$$
$$= c \int_{\mathbb{R}} \frac{d\eta}{\sqrt{((x_1 - y_1)^2 + \eta^2)^2 + 4(2x_2 - 2y_2 + \eta(x_1 + y_1))^2}}.$$

Higher step groups and Grushin type operators

Example: Consider the Engel group $\mathcal{E}_4$ as a matrix group

$$\mathcal{E}_4 = \left\{ \begin{pmatrix} 1 & x & \frac{x^2}{2} & z \\ 0 & 1 & x & w \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \; : \; x, y, w, z \in \mathbb{R} \right\} \subset \mathbb{R}^{4\times 4}.$$

The corresponding Lie algebra $\mathfrak{e}_4$ has the following bracket relations:

$$[X, Y] = W \qquad \text{and} \qquad [X, W] = Z.$$

A 3-step Carnot group

The Engel group $\mathcal{E}_4$ is the lowest dimensional Carnot group of step 3.

Calculate the left-invariant vector fields X and Y on $\mathcal{E}_4$ [3].

$$X := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial w} + \left(\frac{w}{2} - \frac{xy}{12}\right)\frac{\partial}{\partial z},$$
$$Y := \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial w} - \frac{x^2}{12}\frac{\partial}{\partial z}.$$

Lemma

The vector fields X and Y are skew-symmetric on $\mathcal{E}_4$. They span a bracket generating distribution:

$$\mathcal{H} := \text{span}\Big\{X, Y\Big\}.$$

Since $W = [X, Y]$ and $Z = [X, W] = [X, [X, Y]]$.

$$\Delta_{\text{sub}}^{\mathcal{E}_4} = -\frac{1}{2}\Big\{X^2 + Y^2\Big\} = \textit{Sub-Laplacian.}$$

[3] Recall: one uses the Baker-Campbell-Hausdorff formula

Consider the sub-group

$$\mathcal{N} = \{sX + tW \ : \ s, t \in \mathbb{R}\} \cong \mathbb{R}^2$$

of $\mathcal{E}_4 \cong \mathfrak{e}_4$. One obtains a fiber bundle

$$\rho : \mathcal{E}_4 \longrightarrow \mathcal{N} \backslash \mathcal{E}_4 \cong \mathbb{R}^2, \quad \text{where} \quad \rho(x, y, w, z) = \left(x, z + \frac{xw}{2} + \frac{yx^2}{6} \right).$$

Observation

The vector fields X and Y descend via $d\rho$ to $\mathcal{N} \backslash \mathcal{E}_4$. We obtain the Grushin type operator

$$\mathcal{G} = -d\rho(X)^2 - d\rho(Y)^2 = -\frac{\partial^2}{\partial u^2} + \frac{u^4}{4} \frac{\partial^2}{\partial v^2}.$$

Perform a partial Fourier transform with respect to the variable v. We obtain a family of operators on $\mathbb{R}$

$$\mathcal{L}_\eta := -\frac{\partial^2}{\partial u^2} + \frac{u^4}{4}\eta^2 = \text{"quartic oscillator"} \text{ if } \eta \neq 0.$$

These operators are elliptic if $\eta \neq 0$.

Calculating the heat kernel

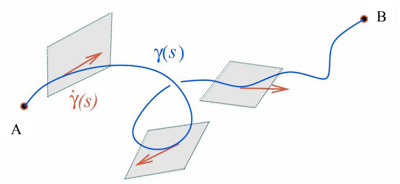
- We could obtain the heat kernel of $\mathcal{G}$ from the heat kernel of $\Delta_{\text{sub}}^{\mathcal{E}_4}$ via a fiber integration:

  More precisely: Let $\Phi : \mathcal{N} \times (\mathcal{N} \setminus \mathcal{E}_4) \longrightarrow \mathcal{E}_4$, be a trivialization of the bundle [a] then:

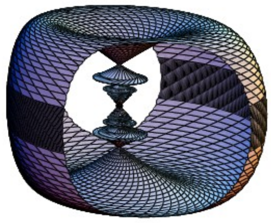
$$K^{\mathcal{G}}(t, \rho(x), y) = \int_{\mathbb{R}^2} K^{\Delta_{\text{sub}}^{\mathcal{E}_4}}(t, x, \Phi(u, y))\, du.$$

[a] trivialization means: $\rho \circ \Phi(x, y) = x$.

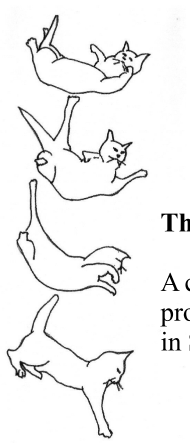
Thank you for your attention!



Distribution and horizontal curve



Front of SR geodesics at time T
(picture by: U. Boscain, D. Barilari)



The falling cat:

A connectivity problem in SR geometry