Ultra-hyperbolic operators on pseudo H-type groups

5. lecture

"Singular Integrals on nilpotent Lie groups and related topics"

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Outline

1. Pseudo H-type Lie groups
2. The heat kernel of the sub-Laplacian and a change of variables
3. The fundamental solution of the UH operator $\Delta_{r,s}$: case $r = 0$
4. Invertibility and local solvability in the case $r > 0$.

Pseudo-H-type Lie algebras

Let $r, s \in \mathbb{N}_0$ and consider $\mathbb{R}^{r,s} = \mathbb{R}^{r+s}$ with bilinear form

$$\langle x, y\rangle_{r,s} = \sum_{i=1}^{r} x_i y_i - \sum_{j=1}^{s} x_{r+j} y_{r+j}.$$

Consider $q_{r,s}(x) := \langle x, x\rangle_{r,s}$ and define

$$C\ell_{r,s} := \textit{Clifford algebra generated by } (\mathbb{R}^{r,s}, q_{r,s}).$$

Clifford module

Let V be a Clifford module, i.e. V is a real vector space with Clifford module action

$$J : C\ell_{r,s} \times V \to V : J_z = J(z, \cdot) : V \to V.$$

This means $J_z J_{z'} + J_{z'} J_z = -2\langle z, z'\rangle_{r,s} I$ for all $z, z' \in \mathbb{R}^{r,s}$.

Pseudo-H-type algebras

Assume: V carries a non-degenerate symmetric bilinear form $\langle \cdot, \cdot\rangle_V$

Definition

We call the module $(V, \langle \cdot, \cdot\rangle_V)$ admissible if

$$\langle J_z X, J_z Y\rangle_V = \langle z, z\rangle_{r,s} \langle X, Y\rangle_V,$$
$$\langle J_z X, Y\rangle_V = -\langle X, J_z Y\rangle_V,$$
$$J_z^2 = -\langle z, z\rangle_{r,s} \mathrm{Id}.$$

Note: the conditions are not independent.

Lemma

If $s > 0$, then $(V, \langle \cdot, \cdot\rangle_V)$ has positive and negative definite subspaces of the same dimension. In particular, $\dim V$ is even.

(Pseudo) H-type algebras

Define a Lie bracket $[\cdot,\cdot] : V \times V \to \mathbb{R}^{r,s}$ through the relations:

$$\langle J_z X, Y \rangle_V = \langle z, [X,Y] \rangle_{r,s}, \qquad z \in \mathbb{R}^{r,s},\ X, Y \in V.$$

Definition

Let V be an admissible $C\ell_{r,s}$- module. With the bracket $[\cdot,\cdot]$ and the center $\mathbb{R}^{r,s}$ the sum

$$\mathcal{N}_{r,s} := V \oplus \mathbb{R}^{r,s}$$

defines a step-2 nilpotent Lie algebra called pseudo H-type algebra.

Definition

The connected, simply connected Lie group $G_{r,s}$ with Lie algebra $\mathcal{N}_{r,s}$ is called pseudo H-type Lie group.

Example: Heisenberg algebra $\mathfrak{h}_{2n+1}$

- Let $z \in \mathbb{R}^{0,1}$ such that $\langle z, z \rangle_{0,1} = -1$.
- Let $V = \mathbb{R}^n \times \mathbb{R}^n$ with basis $(v_1, \cdots, v_n, w_1, \cdots, w_n)$,
- Define a non-degenerate bilinear form on V via:

$$\langle v_i, v_j \rangle_V := \delta_{ij}, \quad \langle w_i, w_j \rangle_V := -\delta_{ij} \quad \textit{and} \quad \langle v_i, w_j \rangle_V := 0.$$

- Define a Clifford module action $J_z : V \to V$ for $z \in \mathbb{R}^{0,1}$ via

$$J_z v_i = w_i \quad \textit{and} \quad J_z w_i = v_i \quad (i, j = 1, \cdots, n).$$

Bracket relations:

$$\langle [v_i, w_i], z \rangle_{0,1} = \langle J_z v_i, w_i \rangle_V = \langle w_i, w_i \rangle = -1.$$

Conclusion

Non-trivial commutator relations: $[v_i, w_i] = z$, i.e., $\mathfrak{h}_{2n+1} \cong \mathcal{N}_{0,1}$.

What is known? - Some References

Positive definite case $(r, s) = (r, 0)$ (Heisenberg type Lie algebras):

A. Kaplan, *Fundamental solution for a class of hypo-elliptic PDE generated by composition of quadric forms*, Trans. Amer. Math. Soc. 258 (1980) no. 1. 147-153.

General case (r, s), $r, s \in \mathbb{N}_0$:

P. Ciatti, *Scalar products on Clifford modules and pseudo-H-type Lie algebras*, Ann. Mat. Pura Appl. 178 (4) (2000), 1-31.

(Standard) Lattices:

K. Furutani, I. Markina, *Existence of lattice on general H-type groups*, J. Lie Theory 24, 979-1011, (2014).

Classification of Lie algebras:

K. Furutani, I. Markina, *Complete classification of pseudo-H-type Lie algebras: I and II*, **Part I:** Geom. Dedicata, 190, 23-51, (2017).

Complete integrability of the bicharacteristic flow ...

Left-invariant vector fields and ultra-hyperbolic operator

Let $\mathcal{N}_{r,s} = V \oplus \mathbb{R}^{r,s}$ be a pseudo H-type algebra. Consider

$$\exp : \mathcal{N}_{r,s} \to (G_{r,s}, *) = \text{pseudo } H\text{-type Lie group}$$

Some standard facts:

- The exponential is a diffeomorphism and allows to identify the Lie algebra $\mathcal{N}_{r,s}$ with the Lie group $G_{r,s}$.
- Product on $\mathcal{N}_{r,s} \cong G_{r,s}$ via Baker-Campbell-Hausdorff formula

$$\exp(X) * \exp(Y) = \exp\Big(X + Y + \frac{1}{2}[X, Y]\Big).$$

- On $\mathcal{N}_{r,s}$ put the scalar product (not necessarily positive def.)

$$\langle x + z, x' + z' \rangle_{\mathcal{N}_{r,s}} = \langle x, x' \rangle_V + \langle z, z' \rangle_{r,s} \quad x + z, x' + z' \in V \oplus \mathbb{R}^{r,s}.$$

Extend it to a left-invariant pseudo Riemannian metric on $G_{r,s}$.

Left-invariant vector fields

Let $s > 0$:

$$\mathcal{X} := [\underbrace{X_1, \cdots X_n}_{\langle X_j, X_j \rangle_V = 1}, \underbrace{X_{n+1}, \cdots, X_{2n}}_{\langle X_j, X_j \rangle_V = -1}] = \text{basis of } V \cong \mathbb{R}^{2n}$$

$$\mathcal{Z} := [Z_k \,:\, k = 1, \cdots, r+s] = \text{basis of } \mathbb{R}^{r+s} = \text{center of } \mathcal{N}_{r,s}.$$

Identify $\mathcal{X}$, $\mathcal{Z}$ with left-invariant vector fields [1] on $G_{r,s} \cong \mathbb{R}^{2n+r+s}$,

$$X_j := \frac{\partial}{\partial x_j} + \sum_{m=1}^{2n} \sum_{k=1}^{r+s} a_{mj}^k x_m \frac{\partial}{\partial z_k}, \quad \text{and} \quad Z_k := \frac{\partial}{\partial z_k}.$$

Structure constants: (a_{mj}^k) are defined via the equation:

$$[X_m, X_j] = 2 \sum_{k=1}^{r+s} a_{mj}^k Z_k, \qquad m, j \in \{1, \ldots, 2n\}.$$

[1] e.g., $[X_j f](g) = \frac{d}{dt} f(g * e^{tX_j})_{|t=0}$.

Ultra-hyperbolic operator

Definition

Let $r, s \in \mathbb{N}_0$ and $s > 0$. We call

$$\Delta_{r,s} := \sum_{j=1}^{n} X_j^2 - X_{j+n}^2$$

an ultra-hyperbolic operator associated to the Lie algebra $\mathcal{N}_{r,s}$. [a]

[a] Due to the similarity of $\Delta_{r,s}$ with the classical ultra-hyperbolic operator $\mathcal{L} = \sum_{j=1}^{n} \partial_{x_j}^2 - \partial_{x_{j+n}}^2$.

Example: Let $\mathbb{H}_3 = \mathcal{G}_{0,1}$ be the 3-dimensional Heisenberg group.

$$\Delta_{0,1} = \left(\frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial z} \right)^2 - \left(\frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial z} \right)^2.$$

Aim of the talk

(a) Characterize the pairs (r, s) for which the ultra-hyperbolic operators $\Delta_{r,s}$ admits an inverse (= fundamental solution).

Remark: The operator $\Delta_{r,s}$ is second order and neither with constant coefficients nor biinvariant under the group action.

(b) Derive a class of fundamental solutions of $\Delta_{r,s}$ in the space of tempered distributions, whenever the existence is guaranteed:

Find $K \in \mathcal{S}'(\mathbb{R}^{2n+r+s})$ explicitly such that

$$\Delta_{r,s} K = \delta_0.$$

(c) Determine pairs (r, s) such that $\Delta_{r,s}$ is locally solvable.

Previous results: invertibility of left-invariant operators

Let $\mathbb{H}_{2n+1}$ be **Heisenberg group** with Lie algebra $\mathfrak{h}_{2n+1}$ and basis

$$V_1, \cdots, V_{2n}, U \quad \textit{such that} \quad [V_j, V_{j+n}] = U \quad j = 1, \cdots, n.$$

- (J. Tie, 2007): Let $\alpha \in \mathbb{C}$ and consider:

$$\Box_\alpha := \sum_{j=1}^{n} \left(V_j^2 - V_{j+n}^2\right) - 2i\alpha U.$$

  In coordinates $U = \frac{\partial}{\partial x_0}$ and with $a_j > 0$

$$V_j := \begin{cases} \frac{\partial}{\partial x_j} - 2a_j x_{j+n} \frac{\partial}{\partial x_0}, & \textit{if} \quad j = 1, \cdots, n \\ \frac{\partial}{\partial x_j} + 2a_j x_{j-n} \frac{\partial}{\partial x_0}, & \textit{if} \quad j = n+1, \cdots, 2n. \end{cases}$$

- (MÃ¼ller/ Ricci, '92): $(a_{ij}) = (a_{ij})^t \in \mathbb{R}(2n)$ and $(s_{ij}) \in \mathrm{sp}(n, \mathbb{R})$:

$$\Box_\alpha^A := \sum_{i,j=1}^{2n} a_{ij} V_i V_j - 2i\alpha U \quad \textit{and} \quad \sum_{i,j=1}^{2n} s_{ij} V_i V_j + 2TU \quad \textbf{(biinvariant)}$$

From the sub-Laplacian to the ultra-hyperbolic operator

With (r,s) as before consider the uh-operator

$$\Delta_{r,s} = \sum_{j=1}^{n} \left\{ \frac{\partial}{\partial x_j} + \sum_{m=1}^{2n} \sum_{k=1}^{r+s} a_{mj}^{k} x_m \frac{\partial}{\partial z_k} \right\}^2 - \left\{ \frac{\partial}{\partial x_{j+n}} + \sum_{m=1}^{2n} \sum_{k=1}^{r+s} a_{m,j+n}^{k} x_m \frac{\partial}{\partial z_k} \right\}^2 .$$

Lemma

Let ξ, η be the dual variables to x and z. $\Delta_{r,s}$ has the symbols:

$$\sigma(\Delta_{r,s})(x,z,\xi,\eta) = -P(\xi) - \frac{\langle \eta, \eta \rangle_{r,s}}{4} P(x) + x^T \rho(\eta)\xi.$$

Here we put $\Omega(\eta) = \eta_1(a_{ij}^1) + \cdots + \eta_{r+s}(a_{ij}^{r+s}) \in \mathbb{R}^{2n\times 2n}$ and

$$P(x) := \sum_{j=1}^{n} x_j^2 - x_{j+n}^2 \qquad \text{and} \qquad \tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

$$\rho(\eta) := -2\Omega(\eta)\tau \in \mathbb{R}^{2n\times 2n}.$$

A change of coordinates

In $\sigma(\Delta_{r,s})(x,z,\xi,\eta)$ change from real to complex coordinates:

With the $x = (x_+, x_-) \in \mathbb{R}^n \times \mathbb{R}^n$ and $z = (z_+, z_-) \in \mathbb{R}^{r,s}$ put:

$$(*) \begin{cases} y_+ &= -ix_+, \\ y_- &= x_-, \\ w_+ &= z_+, \\ w_- &= -iz_-. \end{cases}$$

Let $(\zeta_+, \zeta_-, \vartheta_+, \vartheta_-)$ be the variables dual to (y_+, y_-, w_+, w_-):

Observation:

$(*)$ transforms the symbol $\sigma(\Delta_{r,s})(x,z,\xi,\eta)$ into the symbol of a sub-Laplacian Δ_{sub} of a step-2 nilpotent Lie group:

$$\sigma(\Delta_{r,s})(y,w,\zeta,\vartheta) = \sigma(\Delta_{\text{sub}})(y,w,\zeta,\vartheta) \;\; i.e. \;\; \Delta_{\text{sub}} = -\sum_{j=1}^{2n} Y_j^2.$$

How to find the inverse of $\Delta_{r,s}$?

(Naiv) Strategy:

1. Perform a formal change of variables in the symbol of the uh-operator to obtain the symbol of a sub-Laplacian Δ_{sub} on an H-type group (structure constants are obtained explicitly).
2. Integrate the time-variable in the well-know heat kernel of Δ_{sub} to obtain a fundamental solution of Δ_{sub}.
3. Formally reverse the change of variables in the fundamental solution of Δ_{sub} and interpret as a distribution.

Hope:

- This recipe leads to a meaningful distribution, which then rigorously can be shown to be fundamental solution of $\Delta_{r,s}$.
- If 3. fails, then there is no fundamental solution exists at all.

Fundamental solution of the sub-Laplacian

Let Δ_{sub} denote the sub-Laplacian

$$\Delta_{\text{sub}} = -\sum_{j=1}^{2n} Y_j^2 \quad \textit{where} \quad Y_j = \frac{\partial}{\partial y_j} + \left(\Theta\left(\frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial w_{r+s}}\right) y\right)_j$$

and consider the sub-elliptic heat operator

$$\frac{\partial}{\partial t} + \frac{1}{2}\Delta_{\text{sub}} \quad \textit{on} \quad \mathbb{R}_+ \times \mathbb{R}^{2n+r+s}.$$

Heat kernel of Δ_{sub}:

The heat kernel

$$k : (0,\infty) \times \mathbb{R}^{2n+r+s} \to \mathbb{R}$$

is defined by the conditions:

1. $(\frac{\partial}{\partial t} + \frac{1}{2}\Delta_{\text{sub}})k = 0$,
2. $\lim_{t\downarrow 0} k(t,\cdot) = \delta_0$ (in the sense of distributions).

Heat kernel of the sub-Laplacian

Based on the hypo-ellipticity of Δ_{sub} the heat kernel exists. It is known explicitly:

Theorem (Beals, Gaveau, Greiner, Furutani)

The heat kernel of Δ_{sub} has the form:

$$k_t(y,w) = \frac{1}{(2\pi t)^{n+r+s}} \int_{\mathbb{R}^{r+s}} e^{-\frac{f(y,w,\vartheta)}{t}} W(|\vartheta|)\, d\vartheta,$$

where

$$W(|\vartheta|) = \sqrt{\det \frac{\Theta(i\vartheta)}{\sinh(\Theta(i\vartheta))}} = \left(\frac{\frac{|\vartheta|}{2}}{\sinh\left(\frac{|\vartheta|}{2}\right)}\right)^n, \quad \textit{(volume element)},$$

$$f(y,w,\vartheta) = i\langle \vartheta, w\rangle + \frac{|\vartheta|}{4}\coth\left(\frac{|\vartheta|}{2}\right)\sum_{j=1}^{2n} y_j^2, \quad \textit{(action function)}.$$

From the heat kernel to the fundamental solution

Expectation: A fundamental solution to Δ_{sub} is given by:

$$K_{\text{sub}}(y,w) = \int_0^\infty k_{2t}(y,w)dt.$$

Formal calculation: In fact:

$$\Delta_{\text{sub}} K_{\text{sub}} = \int_0^\infty \Delta_{\text{sub}} k_{2t}\, dt = -\int_0^\infty \frac{\partial}{\partial t} k_{2t}\, dt = -\Big[k_{2t}\Big]_0^\infty = \delta_0.$$

Hence the previous theorem suggests the following expression of a fundamental solution of Δ_{sub}:

$$K_{\text{sub}}(y,w) = \frac{1}{(4\pi)^{p+1}} \int_0^\infty \int_{\mathbb{R}^{r+s}} \frac{1}{t^{p+1}} \exp\left\{-\frac{1}{2t}\left[i\langle\vartheta,w\rangle + \frac{|\vartheta|}{4}\coth\left(\frac{|\vartheta|}{2}\right)\sum_{j=1}^{2n} y_j^2\right]\right\} W(|\vartheta|)\, d\vartheta\, dt,$$

where $p := n + r + s - 1$.

A fundamental solution of $\Delta_{0,s}$ in the case $r = 0$

Reversing the change of variables, changing the path of integration and passing to the Fourier picture gives:

Theorem (W.B. A. Froehly, I. Markina, J. Tie)

With $\vartheta \neq 0$ and $P(\xi) = \sum_j \xi_j^2 - \xi_{j+n}^2$ consider the kernel:

$$q(\xi, \vartheta) := \frac{i}{(2\pi)^{n+\frac{s}{2}}} \int_0^\infty \frac{1}{|\vartheta| \cosh^n t} \exp\left\{ i \frac{\tanh t}{|\vartheta|} \cdot P(\xi) \right\} dt.$$

With the Fourier transform $\mathcal{F}$ and $\varphi \in \mathcal{S}(\mathbb{R}^{2n+s})$ put:

$$K_{0,s}(\varphi) = \int_{\mathbb{R}^{2n+s}} q(\xi, \vartheta) \big[\mathcal{F}\varphi\big](\xi, \vartheta)\, d\xi.$$

Then $K_{0,s}$ is a tempered distribution and it defines a fundamental solution of $\Delta_{0,s}$, i.e.,

$$K_{0,s}(\Delta_{0,s}\varphi) = \varphi(0), \qquad \text{for all} \qquad \varphi \in \mathcal{S}(\mathbb{R}^{2n+s}).$$

A class of fundamental solutions

Problem: *Is the fundamental solution $K_{0,s}$ of the ultra-hyperbolic operator $\Delta_{0,s}$ where $s > 0$ unique?*

Recall:

$$K_{0,s}(\varphi) = \int_{\mathbb{R}^{2n+s}} q(\xi, \vartheta) \big[\mathcal{F}\varphi\big](\xi, \vartheta)\, d\xi\, d\vartheta,$$

where

$$q(\xi, \vartheta) := \frac{i}{(2\pi)^{n+\frac{s}{2}}} \int_0^\infty \frac{1}{|\vartheta| \cosh^n t} \exp\left\{ i \frac{\tanh t}{|\vartheta|} \cdot P(\xi) \right\} dt.$$

Observation

The kernel $q(\xi, \vartheta)$ is of the form

$$q(\xi, \vartheta) = a\Big(P(\xi), |\vartheta|\Big) \quad \text{with} \quad P(\xi) = \sum_{j=1}^n \xi_j^2 - \xi_{j+n}^2,$$

where $a = a(v, w) \in C^\infty\big((\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})\big)$.

A class of fundamental solutions

Lemma: Let $\widetilde{K}_{0,s} \in \mathcal{S}'(\mathbb{R}^{2n+s})$ be a fundamental solution of $\Delta_{0,s}$ s.t.

(a) For $\varphi \in \mathcal{S}(\mathbb{R}^{2n+s})$ consider integral transforms of the type: [2]

$$\widetilde{K}_{0,s}(\varphi) = \int_{\mathbb{R}^{2n+s}} \widetilde{q}(\xi,\vartheta)\big[\mathcal{F}\varphi\big](\xi,\vartheta)\, d\xi\, d\vartheta$$

(b) The kernel $\widetilde{q}(\xi,\vartheta)$ is of the form

$$\widetilde{q}(\xi,\vartheta) = a\big(P(\xi),\vartheta\big) \quad with \quad a \in C^{\infty}\big(\mathbb{R}^{*} \times (\mathbb{R}^{s} \setminus \{0\})\big).$$

ODE:

Then the function $f = a(\cdot,\vartheta)$ for each ϑ is a solution of the ODE:

$$\big(2\pi\big)^{-\frac{2n+s}{2}} = -vf(v) - n|\vartheta|^2 \cdot f'(v) - |\vartheta| \cdot vf''(v).$$

[2]with $\text{supp}(\mathcal{F}\varphi) \subset \{(\xi,\vartheta) \;:\; P(\xi) \neq 0; \;\; \vartheta \neq 0\}$

A class of fundamental solutions

$$\textbf{ODE}: \quad \big(2\pi\big)^{-\frac{2n+s}{2}} = -vf(v) - n|\vartheta|^2 \cdot f'(v) - |\vartheta| \cdot vf''(v).$$

General solution of the ODE

$$\tilde{a}(v,\vartheta) = \frac{i \cdot \sqrt{\pi} \cdot \Gamma\left(\frac{n}{2}\right)}{2(2\pi)^{n+s/2}|\vartheta|}\left(\frac{2|\vartheta|}{P(\xi)}\right)^{\frac{n-1}{2}}\Big\{c_1(\vartheta)J_{\frac{n-1}{2}}\left(\frac{P(\xi)}{|\vartheta|}\right) + c_2(\vartheta)Y_{\frac{n-1}{2}}\left(\frac{P(\xi)}{|\vartheta|}\right) + +i\mathbf{H}_{\frac{n-1}{2}}\left(\frac{P(\xi)}{|\vartheta|}\right)\Big\}.$$

Here $c_1, c_2 : \mathbb{R}^s \to \mathbb{C}$ are measurable functions and

- $J_{\frac{n-1}{2}}$ and $Y_{\frac{n-1}{2}}$ are Bessel functions of the 1^{st} and 2^{nd} kind.
- $\mathbf{H}_{\frac{n-1}{2}}$ is called Struve function.

Remarks

- $J_{\frac{n-1}{2}}$ and $Y_{\frac{n-1}{2}}$ solve the homogeneous Bessel equation:

$$0 = \left\{ v^2 - \left(\frac{n-1}{2}\right)^2 \right\} h(v) + vh'(v) + z^2 h''(v).$$

- The Struve function $\mathbf{H}_{\frac{n-1}{2}}$ solves the inhomogeneous Bessel equation:

$$\frac{4\left(\frac{v}{2}\right)^{\frac{n+1}{2}}}{\sqrt{\pi}\cdot\Gamma\left(\frac{n}{2}\right)} = \left\{ v^2 - \left(\frac{n-1}{2}\right)^2 \right\} h(v) + vh'(v) + z^2 h''(v).$$

- The choice $c_1 \equiv 1$ and $c_2 \equiv 0$ gives the previous fundamental solution obtained via the heat kernel of the sub-Laplacian.
- **Not all** the solutions of the ODE above induce a fundamental solution of $\Delta_{0,s}$ via the above integral transform.

  Problem: Non-integrable singularities on the cone $P(\xi) = 0$.

A class of fundamental solutions

We only consider the special case $c_2 \equiv 0 : \mathbb{R}^s \to \mathbb{C}$ and put:

$$q_{0,s}^{\lambda,\mu}(\xi,\vartheta) = \frac{i}{(2\pi)^{n+s/2}|\vartheta|} \int_0^1 (1-\rho^2)^{\frac{n-2}{2}} \left\{ \lambda(\vartheta) e^{i\frac{P(\xi)}{|\vartheta|}\rho} - \mu(\vartheta) e^{-i\frac{P(\xi)}{|\vartheta|}\rho} \right\} d\rho,$$

where $\lambda, \mu : \mathbb{R}^s \to \mathbb{C}$ are measurable functions.

Theorem (W.-B., A. Froehly, I. Markina)

For bounded measurable functions $\lambda, \mu : \mathbb{R}^s \to \mathbb{C}$ with $\lambda + \mu \equiv 1$

$$K_{0,s}^{\lambda,\mu}(\varphi) := \int_{\mathbb{R}^{2n+s}} q_{0,s}^{\lambda,\mu}(\xi,\vartheta)\big[\mathcal{F}\varphi\big](\xi,\vartheta) d(\xi,\vartheta) dt$$

defines a fundamental solution of $\Delta_{0,s}$ in $\mathcal{S}'(\mathbb{R}^{2n+s})$.

Example: $2n+1$-dimensional Heisenberg group

We rewrite the above fundamental solution for $G_{0,1} = \mathbb{H}_{2n+1}$.

Let $s = 1$ and choose $\lambda : \mathbb{R} \to [0,1]$ as an indicator function:

$$\lambda(\vartheta) := \begin{cases} 1, & \text{if } \vartheta \geq 0, \\ 0, & \text{if } \vartheta < 0. \end{cases} \qquad \textit{and put} \qquad \mu(\vartheta) := 1 - \lambda(\vartheta).$$

We can simplify (= solve the Fourier transform) the form:

$$K_{0,1}^{\lambda,\mu}(\varphi) = \frac{1}{(4\pi i)^n} \int_0^\infty \frac{1}{\sinh^n t} \cdot \frac{\partial^{n-1}\varphi}{\partial z^{n-1}} \left(x, -\frac{P(x)}{4} \coth t \right) dxdt.$$

Remarks

- This distribution vanishes in $\{(x,z) \in \mathbb{R}^{2n+1} : 4|z| < |P(\xi)|\}$.
- Special case of a f.s. due to F. Müller and F. Ricci (1992).

Fundamental solution of $\Delta_{r,s}$ in the case $r > 0$

Problem: *If $r > 0$ then the formal change of variables in the f.s. of Δ_{sub} does not define a distribution.*

Consider $\Delta_{r,s}$ on $\mathcal{S}(\mathbb{R}^{2n+r+s})$ after Fourier transform:

$$\mathcal{F} \circ \Delta_{r,s} = \mathcal{G}_{r,s} \circ \mathcal{F}.$$

This defines an operator $\mathcal{G}_{r,s}$:

The operator $\mathcal{G}_{r,s}$

$$\mathcal{G}_{r,s}\varphi = -P\varphi + \frac{|\eta_+|^2 - |\eta_-|^2}{4} \mathcal{L}\varphi - 2i \left\langle \Omega(\eta)\tau\xi, \nabla_\xi \varphi \right\rangle,$$

where

$$\mathcal{L} = \sum_{j=1}^n \frac{\partial^2}{\partial \xi_j^2} - \frac{\partial^2}{\partial \xi_{j+n}^2} = \text{"classical ultra-hyperbolic operator"}.$$

Fundamental solution of $\Delta_{r,s}$ in the case $r > 0$

We prove:

Theorem

Let $r > 0$, then there is a non-trivial, non-negative valued function $\psi \in \mathcal{S}(\mathbb{R}^{2n+r+s})$ in the kernel of the operator $\mathcal{G}_{r,s}$.

From this we obtain our main result:

Theorem (W.-B., A. Froehly, I. Markina)

Let $r > 0$, then the ultra-hyperbolic operator $\Delta_{r,s}$ does not have a fundamental solution in $\mathcal{S}'(\mathbb{R}^{2n+r+s})$.

Proof: Let $r > 0$ and assume that $\Delta_{r,s}$ admits a fundamental solution $K_{r,s} \in \mathcal{S}'(\mathbb{R}^{2n+r+s})$. Then for all $\psi \in \mathcal{S}(\mathbb{R}^{2n+r+s})$.

$$\Delta_{r,s}K_{r,s}(\psi) = K_{r,s}\Big(\Delta_{r,s}\psi\Big) = \delta_0\psi = \psi(0).$$

Choose a non-negative valued function in the kernel of $\mathcal{G}_{r,s}$

$$0 \neq \psi \in \mathcal{S}(\mathbb{R}^{2n+r+s}) \quad s.t. \quad \mathcal{G}_{r,s}\psi = 0.$$

Fundamental solution of $\Delta_{r,s}$ in the case $r > 0$

Proof (continued): Since

$$\Delta_{r,s} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{G}_{r,s}$$

we have

$$\begin{aligned} K_{r,s}\big(\mathcal{F}^{-1} \circ \mathcal{G}_{r,s}\psi\big) &= K_{r,s}\big(\Delta_{r,s} \circ \mathcal{F}^{-1}\psi\big) \\ &= [\mathcal{F}^{-1}\psi](0) = \frac{1}{(2\pi)^{n+\frac{r+s}{2}}} \int_{\mathbb{R}^{2n+r+s}} \psi(\xi, z)d\xi dz > 0. \end{aligned} \tag{1}$$

On the other hand since $\mathcal{G}_{r,s}\psi = 0$:

$$K_{r,s}(\mathcal{F}^{-1} \circ \mathcal{G}_{r,s}\psi) = 0. \tag{2}$$

(1) and (2) are in contradiction. Hence $K_{r,s}$ cannot exist if $r > 0$.

□

On the local solvability of $\Delta_{r,s}$

Let L be a left-invariant differential operator on $G_{r,s}$.

Definition

We call L *locally solvable* at $x_0 \in G_{r,s}$ if there is an open neighborhood U of x_0 such that

$$LC^\infty(U) \supset C_0^\infty(U),$$

where $C_0^\infty(U)$:=compactly supported smooth functions.

Remarks:

(a) Left-invariant implies: L is locally solvable at one point if and only if it is locally solvable at any point of $G_{r,s}$.

(b) Because of (a) call L locally solvable if L is locally solvable at $x_0 = 0$.

Fundamental solution in $\mathcal{D}'(\mathbb{R}^{2n+r+s})$?

Question

Does $\Delta_{r,s}$ with $r > 0$ have a f.s. in the larger space $\mathcal{D}'(\mathbb{R}^{2n+r+s})$?

$G_{r,s}$ is homogeneous, i.e. there is a family $\{\delta_\rho\}$ of dilations on $G_{r,s}$:

$$\delta_\rho : G_{r,s} \cong \mathbb{R}^{2n+r+s} \to G_{r,s} : \delta_\rho(x,z) := \big(\rho \cdot x, \rho^2 \cdot z\big).$$

Moreover, each map δ_ρ is an automorphisms of $G_{r,s}$:

$$\delta_\rho\Big((x,z) * (y,w)\Big) = \delta_\rho(x,z) * \delta_\rho(y,w).$$

Definition (homogeneous operator)

A left-invariant operator L on $G_{r,s}$ is called homogeneous of degree k if

$$\delta_\rho^* L = \rho^k L \quad \textit{for all} \quad \rho > 0.$$

Ex.: $\Delta_{r,s}$ is homogeneous on $G_{r,s}$ of degree 2.

On the local solvability of $\Delta_{r,s}$

Theorem (F. Battesti)

Let L be left-invariant and homogenous on $G_{r,s}$. Then the following are equivalent:

(a) L *is locally solvable,*

(b) $LC^\infty(G_{r,s}) = C^\infty(G_{r,s})$,

(c) L *has a fundamental solution* $E \in \mathcal{D}'(G_{r,s})$.

Strategy:

We give a negative answer to the above question by proving that $\Delta_{r,s}$ is not locally solvable on $G_{r,s}$ if $r > 0$.

A criterion for non-local solvability

Theorem (D. MÃ¼ller, 1991)

Let L be a left-invariant homogeneous differential operator on a homogeneous, simply connected nilpotent Lie group G with transpose L^τ. Assume there exists a sequence $\{\psi_j\}_{j=1}^\infty$ of Schwartz functions on G

(i) $\psi_j(0) = 1$ *for every* j,

(ii) *For every continuous semi-norm $\|\cdot\|_{(N)}$ on the Schwartz space $\mathcal{S}(G)$ it holds:*

$$\lim_{j\to\infty} \|\psi_j\|_{(N)} \cdot \|L^\tau \psi_j\|_{(N)} = 0.$$

Then L is not locally solvable.

Ex.: If $\psi \in \mathcal{S}(G)$ in the kernel of L^τ with $\psi(0) = 1$ exists, then we may choose the constant sequence $\psi_j = \psi$ for all $j \in \mathbb{N}$.

On the local solvability of $\Delta_{r,s}$

Applying the last criterion to a previous theorem gives:

Theorem (W. Bauer, A. Froehly, I. Markina)

In the case $r > 0$ the ultra-hyperbolic operator $\Delta_{r,s}$ is not locally solvable. In particular, $\Delta_{r,s}$ does not even admit a fundamental solution in the space of Schwartz distributions $\mathcal{D}'(G_{r,s})$ and

$$\Delta_{r,s} C^{\infty}(\mathbb{R}^{2n+r+s}) \subsetneq C^{\infty}(\mathbb{R}^{2n+r+s}).$$

If $r = 0$, then we also have a positive result:

Theorem (W. Bauer, A. Froehly, I. Markina)

The operator $\Delta_{0,s}$ where $s > 0$ are locally solvable and

$$\Delta_{r,s} C^{\infty}(\mathbb{R}^{2n+s}) = C^{\infty}(\mathbb{R}^{2n+s}).$$

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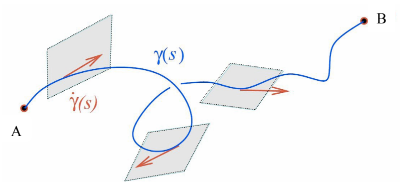
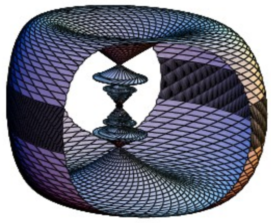
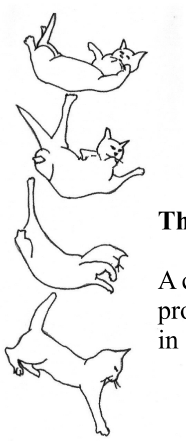
Thank you for your attention!



Distribution and horizontal curve



Front of SR geodesics at time T
(picture by: U. Boscain, D. Barilari)



The falling cat:

A connectivity problem in SR geometry

On the singular support of $K_{0,s}^{1,0}$

With our previous notation consider the fundamental solution $K_{0,s}^{1,0}$:

Theorem

The singular support of $K_{0,s}^{1,0}$ is contained in the cone

$$\mathcal{C} = \Big\{(x,z) \in \mathbb{R}^{2n+s} \, : \, 4|P(x)| \leq |z|\Big\}.$$

Conjecture:

The singular support is contained in the boundary of the cone.

J. Tie, *The inverse of some differential operators on the Heisenberg group*, Comm. Pure Math. 20 (7-8), 1275-1302.