Smooth frames in the orbit of a square-integrable representation of a nilpotent Lie group

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Outline

I. Background and motivation

II. Density theorems for lattices

III. Non-uniform density theorems

IV. Complete systems and overcomplete frames

V. Open problems

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An irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ is *square-integrable* if there exists $g \in \mathcal{H}_\pi \setminus \{0\}$ such that

$$\int_{N/Z} |\langle g, \pi(x) g \rangle|^2 \, d\mu_{N/Z}(x) < \infty,$$

where $\mu_{N/Z}$ denotes Haar measure on N/Z.

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For a (smooth) cross-section $s : N/Z \to N$ of the projection $q : N \to N/Z$, i.e., $q \circ s = \mathrm{id}_{N/Z}$, define

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The pair $(\overline{\pi}, \mathcal{H}_\pi)$ is an irreducible, square-integrable *projective* representation of N/Z, i.e.,

$$\overline{\pi}(xy) = \sigma(x, y)\overline{\pi}(x)\overline{\pi}(y), \quad x, y \in N/Z,$$

for a smooth function $\sigma : N/Z \times N/Z \to \mathbb{T}$, called the *cocycle*.

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Throughout, $\overline{\pi}$ will simply be denoted by π and $G := N/Z$.

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Example (Heisenberg group)

The group $N = \mathbb{R}^{2d+1}$ with group multiplication

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For $\xi \neq 0$, the Schrödinger representation $(\pi_\xi, L^2(\mathbb{R}^d))$ of $N = \mathbb{R}^{2d+1}$ is

$$\pi_\xi(x, y, z)f(t) = e^{2\pi i \xi z} e^{-2\pi i \xi y \cdot t} f(t - x), \quad t \in \mathbb{R}^d.$$

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An associated projective representation of $G = \mathbb{R}^{2d}$ is given by

$$\overline{\pi}_\xi(x, y)f(t) = e^{-2\pi i \xi y \cdot t} f(t - x), \quad t \in \mathbb{R}^d,$$

for $f \in L^2(\mathbb{R}^d)$. The associated cocycle is $\sigma((x, y), (x', y')) = e^{-2\pi i x \cdot y'}$.

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For $g \in \mathcal{H}_\pi \setminus \{0\}$, the orbit

$$\pi(G)g = \{\pi(x)g \; : \; x \in G\}$$

is *complete* in $\mathcal{H}_\pi$, i.e., $\overline{\operatorname{span}}\{\pi(x)g : x \in G\} = \mathcal{H}_\pi$:

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Moreover, it follows that $\pi(G)g$ is *overcomplete*, i.e., it remains complete after removal of an arbitrary element.

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Any $f \in \mathcal{H}_\pi$ admits the integral formula

$$f = \int_G \langle f, \pi(x)g \rangle \pi(x)g \, d\mu_G(x), \tag{2}$$

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Questions:

1. (Perelomov): Conditions on discrete subsets $\Lambda \subseteq G$ such that

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2. (Daubechies, Grossmann, Meyer): Discrete analogues of (2) of the form

   $$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g$$

   for discrete $\Lambda \subseteq G$, [J. Math. Phys., '86].

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A countable family $(\pi(\lambda)g)_{\lambda \in \Lambda}$ is a *frame* for $\mathcal{H}_\pi$ if there exist $A, B > 0$ s.t.

$$A\|f\|_{\mathcal{H}_\pi}^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|_{\mathcal{H}_\pi}^2, \quad \text{for all} \quad f \in \mathcal{H}_\pi.$$

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Fact: $\pi(\Lambda)g$ is a Riesz basis iff $\pi(\Lambda)g$ is a non-overcomplete frame.

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Let $\Gamma \leq G$ be a lattice. The following assertions are equivalent:

(i) *There exists $g \in \mathcal{H}_\pi$ such that $\pi(\Gamma)g$ is complete;*

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If $\pi(\Gamma)g$ is a frame for $\mathcal{H}_\pi$, then

$$S_{g,\Gamma} : \mathcal{H}_\pi \to \mathcal{H}_\pi, \quad f \mapsto \sum_{\gamma \in \Gamma} \langle f, \pi(\gamma)g \rangle \pi(\gamma)g$$

is bounded and invertible. The system $\pi(\Gamma)S_{g,\Gamma}^{-1/2}g$ is a Parseval frame.

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Corollary

Let $\Gamma \leq G$ be a lattice and $g \in \mathcal{H}_\pi$.

(i) *If $\pi(\Gamma)g$ is complete, then $\mathrm{vol}(G/\Gamma)d_\pi \leq 1$.*

(ii) *If $\pi(\Gamma)g$ is a frame and $\mathrm{vol}(G/\Gamma)d_\pi < 1$, then $\pi(\Gamma)g$ is overcomplete.*

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By the orthogonality relations,

$$d_\pi^{-1}\|f\|_{\mathcal{H}_\pi}^2\|g\|_{\mathcal{H}_\pi}^2 = \int_G |\langle f, \pi(x)g\rangle|^2 \, d\mu_G(x) = \int_{G/\Gamma} \sum_{\gamma\in\Gamma} |\langle f, \pi(x\gamma)g\rangle|^2 \, d\mu_G(x)$$

$$= \int_{G/\Gamma} \sum_{\gamma\in\Gamma} |\langle \pi(x)^* f, \pi(\gamma)g\rangle|^2 \, d\mu_G(x).$$

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The frame inequalities yield that $A\,\mathrm{vol}(G/\Gamma) \leq d_\pi^{-1}\|g\|_{\mathcal{H}_\pi}^2 \leq B\,\mathrm{vol}(G/\Gamma)$.

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(i) It may be assumed that $\pi(\Gamma)g$ is a Parseval frame. Therefore,

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where it is used that $\|g\|_{\mathcal{H}_\pi}^4 = |\langle g, g\rangle|^2 \leq \sum_\gamma |\langle g, \pi(\gamma)g\rangle|^2 \leq \|g\|_{\mathcal{H}_\pi}^2$.

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(ii) It may be assumed that $\pi(\Gamma)g$ is an orthonormal system. Thus, $\|g\|_{\mathcal{H}_\pi} = 1$ and one can choose $B = 1$ by Bessel's inequality. Therefore,

$$d_\pi^{-1} = d_\pi^{-1}\|g\|_{\mathcal{H}_\pi}^2 \leq \mathrm{vol}(G/\Gamma),$$

as required. □

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Proof ingredients:

1. Hilbert modules over twisted group von Neumann algebras $\mathrm{vN}(\Gamma, \sigma)$.

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(ii) *If* $\mathrm{vol}(G/\Gamma)d_\pi \geq 1$*, then there is* $g \in \mathcal{H}_\pi$ *s.t.* $\pi(\Gamma)g$ *is a Riesz sequence.*

In particular, if $\mathrm{vol}(G/\Gamma)d_\pi = 1$*, then there is an orthonormal basis* $\pi(\Gamma)g$*.*

Proof ingredients:

1. Hilbert modules over twisted group von Neumann algebras $\mathrm{vN}(\Gamma, \sigma)$.
2. $Z(\Gamma) = \Gamma \cap Z(G)$.

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2. Strict comparison of projections in twisted C^*-algebra $C^*(\Gamma, \sigma)$.

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Outline

I. Background and motivation

II. Density theorems for lattices

III. Non-uniform density theorems

IV. Complete systems and overcomplete frames

V. Open problems

Beurling densities

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The *lower* and *upper Beurling density* of a set $\Lambda \subseteq G$ are defined by

$$D^{-}(\Lambda) := \liminf_{R \to \infty} \inf_{x \in G} \frac{\#(\Lambda \cap B_R(x))}{\mu(B_R(e))}$$

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Example: If $\Gamma \leq G$ is a lattice, then $D^{-}(\Gamma) = D^{+}(\Gamma) = 1/\operatorname{vol}(G/\Gamma)$.

Necessary density conditions

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Let $\mathcal{B}_\pi$ be the collection of vectors $g \in \mathcal{H}_\pi$ such that

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$$\pi(\varphi)h := \int_G \varphi(x)\pi(x)h \, d\mu_G(x), \quad \varphi \in C_c(G), \; h \in \mathcal{H}_\pi,$$

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Theorem (Führ, Gröchenig, Haimi, Klotz, Romero, '16)

Let $\Lambda \subseteq G$ *and* $g \in \mathcal{H}_\pi$.

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Extensions: Similar statements hold for unimodular solvable Lie groups with possibly exponential growth; (Enstad, V., '22) and (Caspers, V. '22).

Stability under dilations

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The group G is called *homogeneous* if its Lie algebra $\mathfrak{g}$ admits automorphisms of the form

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Theorem (Gröchenig, Romero, Rottensteiner, V., '20)

Let G be a homogeous nilpotent Lie group with dilations $(\delta_r)_{r>0}$.
If $g \in \mathcal{H}_\pi^\infty$ and $\pi(\Lambda)g$ is a frame (resp. Riesz sequence), then there exists $\varepsilon > 0$ such that, for all $r \in (1-\varepsilon, 1+\varepsilon)$, the system

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Theorem is an adaption/extension of a corresponding theorem for $G = \mathbb{R}^{2d}$ by Gröchenig, Ortega-Cerdà and Romero [Adv. Math., '14].

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[2] Orthonormal bases $\pi(\Lambda)g$ with $g \notin \mathcal{H}_\pi^\infty$ do exist in the orbit of arbitrary nilpotent Lie groups G; cf. Gröchenig and Rottensteiner [J. Funct. Anal., '18] and Oussa [J. Funct. Anal., '19].

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Recall: If $\pi(\Gamma)g$ is complete in $\mathcal{H}_\pi$ for a lattice $\Gamma \leq G$, then

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Density conditions for complete systems

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Theorem (Wang; Appl. Comput. Harmon. Anal. '04)

Let $\Gamma \leq \mathbb{R}^2$ be a lattice with $D(\Lambda) > 1$. For any $\varepsilon > 0$, there exists $g \in L^2(\mathbb{R})$ and a subset $\Lambda \subseteq \Gamma$ with

$$D^-(\Lambda) = 0 \quad \textit{and} \quad D^+(\Lambda) < \varepsilon$$

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Approximate lattices

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A subset $\Lambda \subseteq G$ is a *Delone set* if

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A lattice $\Lambda \leq G$ is an approximate lattice, but also suitable subsets thereof.[3]

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In particular, if $\Lambda \leq G$ is a lattice, then $D^-(\Lambda) \geq d_\pi$.

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Overcompleteness of smooth frames

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By necessary density, a frame $\pi(\Lambda)g$ for $\mathcal{H}_\pi$ with $g \in \mathcal{B}_\pi$ and

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Theorem (Caspers, V., '22)

Suppose $\pi(\Lambda)g$ is a frame for $\mathcal{H}_\pi$ with $g \in \mathcal{H}_\pi^\infty$ and $D^-(\Lambda) > d_\pi$. Then there exists $\Lambda' \subseteq \Lambda$ with $D^-(\Lambda') > 0$ such that $(\pi(\lambda)g)_{\lambda \in \Lambda \setminus \Lambda'}$ is a frame for $\mathcal{H}_\pi$.

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Consequence: Infinitely many elements from $\pi(\Lambda)g$ can be removed yet leave a frame.

Outline

I. Background and motivation

II. Density theorems for lattices

III. Non-uniform density theorems

IV. Complete systems and overcomplete frames

V. Open problems

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1. **Strict density inequalities.** For a non-homogeneous group G: If $\pi(\Lambda)g$ is a frame for $\mathcal{H}_\pi$ with $g \in \mathcal{H}_\pi^\infty$, then

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Open problems

1. **Strict density inequalities.** For a non-homogeneous group G: If $\pi(\Lambda)g$ is a frame for $\mathcal{H}_\pi$ with $g \in \mathcal{H}_\pi^\infty$, then

$$D^-(\Lambda) > d_\pi.$$

2. **Near the critical density.** For every $\varepsilon > 0$ and $g \in \mathcal{H}_\pi^\infty$, there exists $\Lambda \subseteq G$ such that

$$D^+(\Lambda) \leq d_\pi + \varepsilon$$

and $\pi(\Lambda)g$ is a frame for $\mathcal{H}_\pi^\infty$.

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