Spring School

Modern Topics on Analysis on Lie Groups

April 02.–05., 2024

Information, schedule, & abstracts

Scientific Organizers:

- Wolfram Bauer (Hanover, Germany)
- Christian Jäh (Göttingen, Germany)
- Ingo Witt (Göttingen, Germany)

Administrative Organizers:

- Linda Haber (Göttingen, Germany)
- Annalena Wendehorst (Göttingen, Germany)

Contents

1. Schedule

- There will be preparatory lectures about *Nilpotent Lie Groups and -Algebras* as well as some Analysis on those as an introduction to the *Fourier Transform* which will be instrumental in the advanced courses. See Section 2.
- The Q & A sessions are reserved for in-depth questions and discussions on the presented material. You can also use the **Schlauch** (behind the Sitzungszimmer) to have discussions and contemplate the material. The lectures will all be in the **Sitzungszimmer**.
- Social on Tuesday: We will have some snacks, wine and beer as well as non-alcoholic drinks of course. It will be a loose gathering that hopefully facilitates to get to know each other a bit better.
- There will be a sort of conference dinner also at the institute the precise form of which has not yet been determined.

In the schedule, the following abbreviations are used:

- CFK – Clotilde Fermanian Kammerer
- SF – Serena Federico
- DB – Davide Barilari
- AM – Azita Mayeli
- CJ – Christian Jäh
- PT – Participants Talk

Time	April 02	April 03	April 04	April 05
9:00–10:00	CJ	SF	DB	CFK
10:00–10:20	Coffee	Coffee	Coffee	Coffee
10:20–11:20	CJ	DB	SF	DB
11:20–12:20	CJ	CFK	AM	AM
12:20–14:00	Lunch	Lunch	Lunch	Lunch
14:00–15:00	CFK	AM	CFK	
15:00–16:00	AM	Q & A	Q & A	
16:00–16:20	Coffee	Coffee	Coffee	
16:20–17:20	DB	SF	PT	
17:20–18:20	SF		PT	
18:20–	Social		BBQ[1]	

2. Abstract – Introductory Course

We will review some basics around nilpotent Lie groups and algebras as far as they are needed for the advanced courses. We will for instance give some wick review on the Fourier transform as it will feature prominently.
A link to notes and the table of contents will be posted soon.

2.1. **Further introductory material.** There are some introductory lectures available on Youtube, given by one of our speakers Davide Barilari (Padova, Web), concerning the Heisenberg group and its sub-Riemannian geometry:

- Lecture 1, Lecture 2, Lecture 3.

Further, the material presented at our previous school *Singular Integrals on Nilpotent Lie Groups and Related Topics* can be seen as preparatory material. In particular the nice notes from Veronique Fischer (Bath, Web)

- Harmonic Analysis on the Heisenberg Group and Related Topics.

3. Abstracts – Advanced Courses

3.1. Semi-classical analysis on nilpotent graded Lie groups.

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The semi-classical approach aims at analyzing high-frequency phenomena. Via the correspondence principle, it links the quantum world to the classical one, by drawing a connection between the properties of the solution of a PDEs in relation and dynamical features of the symbol of the operator involved in the PDE. For this reason, the semi-classical approach proved to be a powerful tool in different contexts, control theory and spectral geometry for example. We will explain how to define semi-classical tools adapted to the gradation of a nilpotent graded Lie groups: semi-classical pseudodifferential operators, semi-classical measures and wave-packets. We will also discuss applications to spectral geometry on nilmanifolds.

3.2. **Weyl Calculus on graded Lie groups.**

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In this mini-course we will introduce a pseudo-differential calculus on graded nilpotent Lie groups recently developed in collaboration with D. Rottensteiner and M. Ruzhansky. This calculus represents the Weyl calculus on the Heisenberg group and the right candidate Weyl calculus in the general graded group setting. This Weyl quantization, along with the corresponding calculus, will be suitably identified within the family of the so-called $\boldsymbol{\tau}$-quantizations.

Throughout the lectures we will cover the following arguments.

- Preliminaries on the necessary tools on nilpotent Lie groups: representations, Fourier transform, left/right invariant vector fields, Taylor expansion.
- Introduction to the Kohn-Nirenberg quantization on nilpotent groups introduced by M. Ruzhansky and V. Fischer.
- Singular integrals: the fundamental role of kernel estimates.
- $\boldsymbol{\tau}$-quantizations on graded groups and explicit examples in the case of the Heisenberg group.
- $\boldsymbol{\tau}$-calculi on graded Lie groups: asymptotic formulas.
- Invariance properties and the Weyl quantization.
- The Weyl quantization on the Heisenberg group and the candidate in the general graded group setting.

At times, to simplify the exposition, we will restrict the analysis to the case of the Heisenberg group.

3.3. **Strichartz estimates and sub-Riemannian geometry: an introduction.**

(1) 1 Dispersion for Shroedinger. Grushin space, Heisenberg group. No dispersion.
(2) 2 Euclidean Fourier restriction. Relation with Strichartz estimates
(3) 3 Kilrillov theory for Nilpotent groups.
(4) 4 Fourier transform and applications to Carnot groups.

3.4. **Advanced Spatio-Spectral Analysis in Higher Dimensions.**

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These lecture series explore the theoretical foundations and applications of spatio-spectral limiting operators (SSLO) in higher dimensions, extending the study of prolate spheroidal wave functions and their role in time-frequency analysis. The lectures cover the eigenvalue distribution of these operators, providing insights into their behavior and implications in various fields such as signal processing, imaging, and data compression.
Within these lectures, we will focus on the following specific topics:

- **Introduction to Spatio-Spectral Concentration Problem**: An overview of the Heisenberg uncertainty principle and the role of prolate spheroidal wave functions in one-dimensional settings.
- **Introduction to Spatio-Spectral Limiting Operators (SSLO)**: Study of the operators and their spectral analysis.
- **Eigenvalue Distributions of SSLO**: Study of the eigenvalue distribution for SSLO, including asymptotic distribution and clustering behavior of eigenvalues.
- **Wave Packets and Orthonormal Bases**: Techniques for designing wave packets that serve as approximate eigenfunctions of spatio-spectral limiting operators, focusing on their application in higher dimensions.
- **Quantitative Bounds and Applications**: Analysis of quantitative bounds on eigenvalue distributions within given spatial and frequency domains, with a focus on signal manipulations.

Relevant reading: [1], [2], [3], [4], [4]

References.

[1] Arie Israel. *The Eigenvalue Distribution of Time-Frequency Localization Operators.* 2015. arXiv: 1502.04404 [math.CA].

[2] Arie Israel and Azita Mayeli. "On the eigenvalue distribution of spatio-spectral limiting operators in higher dimensions". In: *Appl. Comput. Harmon. Anal.* 70 (2024), Paper No. 101620, 28. DOI: 10.1016/j.acha.2023.101620.

[3] Santhosh Karnik, Justin Romberg, and Mark A. Davenport. "Improved bounds for the eigenvalues of prolate spheroidal wave functions and discrete prolate spheroidal sequences". In: *Appl. Comput. Harmon. Anal.* 55 (2021), pp. 97–128. DOI: 10.1016/j.acha.2021.04.002.

[4] H. J. Landau and H. O. Pollak. "Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals". In: *Bell System Tech. J.* 41 (1962), pp. 1295–1336. DOI: 10.1002/j.1538-7305.1962.tb03279.x.

4. Abstracts – Contributed Talks and Posters

4.1. An uncertainty principle regarding the decay of Fourier transform on Heisenberg groups.

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An uncertainty principle due to Ingham provides the best possible decay of the Fourier transform of a compactly supported function on the real line. In this talk, we intend to explore an analogue of this result by investigating the optimal operator-valued decay of the group Fourier transform of compactly supported functions on the Heisenberg group.

Relevant references: [1], [2]

References.

[1] Pritam Ganguly and Sundaram Thangavelu. "Analogues of theorems of Chernoff and Ingham on the Heisenberg group". In: *J. Anal. Math.* 149.1 (2023), pp. 281–305. DOI: 10.1007/s11854-022-0252-1.

[2] Sayan Bagchi et al. "An analogue of Ingham's theorem on the Heisenberg group". In: *Math. Ann.* 387.1-2 (2023), pp. 1073–1104. DOI: 10.1007/s00208-022-02479-5.

4.2. Solving a truncated K-moment problem with Fourier methods.

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Let d be an arbitrary positive integer, $\mathbb{N}_0^d$ be the set of all d-tuples of non-negative integers, and for any $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ let $|\alpha| = \alpha_1 + \cdots + \alpha_d$. If n is another arbitrary positive integer and K a closed subset of $\mathbb{R}^d$, the truncated K-moment problem examines the existence of a non-negative measure μ, supported on K, such that, for a given (real valued) sequence $\{s_\alpha\}_{0\le|\alpha|\le n}$,

$$s_\alpha = \int_K x^\alpha \mathrm{d}\mu(x), \text{ for any } \alpha \in \mathbb{N}_0^d \text{ such that } |\alpha| \le n.$$

The talk is divided in two parts. In the first one, we present a solution when $n = d$ and K is the set of the hypercube vertices $K_d = \{-1, 1\}^d$. Thanks to the group structure of K_d, the solution is based on an approach that uses simple tools from Fourier analysis. In the second part, by using a separation argument from functional analysis, we see an application of the main theorem, from the first part, that gives a description of the set of non-negative polynomials on K_d of degree at most d. The talk is based on a joint work with C. Emary and D. Kimsey at Newcastle University during the academic year 2022-2023.

5. Where to go for lunch?

For the Zentralmensa, you need a guest card to pay as there is no cash payment at the moment. You can get the guest card at the information desk at the mensa (on the first day, someone of us will help you) and you can then charge it on a machine with an EC card.

- Zentralmensa (Google maps). This is the main mesa on the central campus. This is about 20 minutes walk from the institute.[2] The menu can be found here (choose Zentralmensa in the drop-down). The menu is also displayed in English at the mensa.
  In the Café Zentral a selection of hot and cold sandwiches and pizzas as well as sweets is offered.
- Bakery Küster (here or here).
  They have a great selection of hot and cold sandwiches, salads, and other snacks. Not to forget sweets.
  Here is a link to the Cafe menu. (German)
- Bakery Holzofenbäckerei (Google maps).
  Here you can get sandwiches, soup, and some other snacks.
- Vietnamese Restaurant Nam Anh (Vietnamese, Google maps)
  Groner Str. 12, 37073 Göttingen
- Asian Fusion Gamie (Vietnamise/Japanese, Google maps)
  Weender Str. 29, 37073 Göttingen
- Bullerjahn (German/International, Google maps) Markt 9, 37073 Göttingen
- Empanadas Sabrosita (Spanish, Google maps)
  Karspüle 9, 37073 Göttingen

[2] Can be connected with a stroll to Gauß' grave on the way back.

6. Where to go for dinner?

Here is a list of reasonable restaurants:

- Kartoffelhaus (German, Google maps)
  Goetheallee 8, 37073 Göttingen, Menu (in German)
- Vietnamese Restaurant Nam Anh (Vietnamese, Google maps)
  Groner Str. 12, 37073 Göttingen
- Asian Fusion Gamie (Vietnamise/Japanese, Google maps)
  Weender Str. 29, 37073 Göttingen
- Bullerjahn (German/International, Google maps) Markt 9, 37073 Göttingen
- Abessina (Ethiopian restaurant, Google maps)
  Ritterplan 2, 37073 Göttingen

The institute can provide a list with further options and of course we will try to be as helpful in finding something suitable if you ask us.

7. The University and the city of Göttingen

Some sights/museums:

- The University has a lot of collections which you can brows here. In Mathematics, we have in particular a collection of models, calculating machines, integrators, etc. These can in part be viewed in the foyer in front of the room where the school takes place. Further information here.
- Forum Wissen – the new museum of knowledge in Göttingen – aims to make research with objects tangible., (Admission free.)
- Gänselisel on the main market place in front of the old city hall.
  PhD student traditionally bring her, following a successful defence, flowers and give her a kiss.
- Gauß' grave
- Gauß Observatory
- City Cemetery with graves of David Hilbert, Max Born, Otto Hahn, Max Planck and many more.

8. RTG 2491 - Fourier Analysis and Spectral Theory

The Research Training Group (RTG) Fourier Analysis and Spectral Theory is a joint research and graduate education programme funded by the German Science Foundation (DFG). It is based at the Mathematical Institute, University of Göttingen, with participating researchers from the University of Hanover. The initiative started October 1st, 2019 with the opening of ten PhD positions and one postdoctoral position.

The RTG Fourier Analysis and Spectral Theory is taking an interdisciplinary and innovative approach to the classical and powerful machinery modern harmonic and Fourier analysis and spectral theory. We focus on its development in the context of mathematical physics, topology and analytic number theory.

A core theme of the RTG is analysis and spectral geometry on Riemannian manifolds, in particular, locally symmetric spaces or more generally spaces acted on by groups. Besides a topological structure, in many interesting cases they also have some arithmetic or combinatorial structure, and one of the key questions involves the fascinating interplay between the spectral properties of certain associated operators on the one hand, and geometric, topological or arithmetic properties on the other. Some prototypical examples of this interaction featured in this RTG are the spectral theory of Cayley graphs of groups; analytic L^2-invariants, which link harmonic analysis to topology; and the resolvent and scattering theory of geometric differential operators on singular manifolds. A cornerstone at the interface of modern analytic number theory and harmonic analysis is the theory of automorphic forms, viewed as eigenfunctions of a family of operators on a locally symmetric space. Fourier and harmonic analysis also appear prominently in many applications of classical analytic number theory, in the representation theory of Lie groups and groupoids, and in the construction of quantum field theories with microlocal methods.

On the methodological side we, draw from a variety of analytic techniques, such as microlocal analysis, symbolic calculus, trace formulas and Plancherel theory, Fourier analysis in numerous variations, spectral and scattering theory of operators, but also classical analysis such as a careful analysis of oscillatory integrals.