This lecture will be given at the *Georg-August University* in the *Summer Semester 2026* by Christian Jäh. Vorlesungsverzeichnis.

Prerequisites: Analysis I-II, (Analysis on Manifolds), Complex Analysis, Algebra (Rings and Modules).

Helpful: Several Complex Variables I-II and Lectures in which one learns sheaves and categories as well as cohomology theories. Not having had this is not a reason to not hear this course but it will be a bit harder.

Lecture: Rudiments of Complex Analytic Geometry

In complex analytic geometry[1], complex spaces[2] generalize complex manifolds. They are typically treated in the language of locally ringed spaces and sheaf theory, with coherent analytic sheaves playing a central role. [**?**]

Complex spaces become necessary because complex manifolds are too restrictive for several complex variables. One guiding problem is to find an appropriate higher-dimensional analogue of Riemann surfaces. Even in classical algebraic geometry one encounters singular points that cannot look locally like a standard "ball" in $\mathbb{C}^n$; a basic example is the hypersurface

$$w^2 - z_1 z_2 = 0.$$

The origin is singular, and the hypersurface can be viewed as the natural geometric object associated with a multivalued holomorphic expression such as $\sqrt{z_1 z_2}$. Thus, even objects arising directly from holomorphic equations in several variables may fail to be manifolds at some points, and one needs a broader framework.

In this course we will introduce and study that framework. We will use tools from commutative algebra to analyze local properties, and sheaf-theoretic methods to address local-to-global questions. In this sense, the course follows what Remmert called the *algebraization of complex analysis*.

What you need is a solid grasp of the standard notions listed in the prerequisites; everything else will be introduced as needed. Familiarity with

[1] The course has the broader title complex geometry in StudIP.

[2] In some sense one might also want to say that analytic geometry is, as in algebraic geometry, the geometry of analytic sets. Complex spaces are glued together from analytic sets as local models. See for instance nlab and [Abh01].

sheaves will make the beginning easier, but we will develop the necessary sheaf theory in the course. You should expect to work through exercises[3], and to be willing to look up unfamiliar notions as they appear—this is part of working with a technically demanding subject.

The **main reference** for this course will be Grauert & Remmert, *Coherent Analytic Sheaves* [GR84] also [Fis76, Loj91, GR71, GR77]. For a broader view see also [MM10]. Here are some headline topics:

- Complex spaces [GR58]
- Weierstrass division and preparation theorems
- Coherence results (e.g. Oka/Cartan coherence; coherence of ideal sheaves and of $\mathcal{O}_X$ in basic situations)
- Stein spaces, Cartan's Theorems A and B and their consequences
- Dimension theory (local analytic dimension and its relation to Krull dimension of $\mathcal{O}_{X,x}$ [AM16]; basic comparison with algebraic geometry)
- Extension phenomena (Riemann/Hartogs-type extension; selected extension theorems for analytic sets and functions)
- Analytic covering maps (and selected applications)
- Chow's theorem (analytic subvarieties of $\mathbb{CP}^n$ are algebraic) [?]
- Grauert's direct image theorem (proper holomorphic maps and coherence of higher direct images)
- Selected aspects of Serre's GAGA principle (comparison between algebraic and analytic categories in the projective case) [Ser56]

I will provide lecture notes that include prerequisite material to some extent. However, the course will take some courage on your side and the willingness to work. A working knowledge of commutative algebra (Noetherian rings, localization, Nakayama, Krull dimension) is recommended or the relevant facts need to be read on the side; a standard reference is Atiyah–Macdonald. [AM16]

[3] Exercises will not be marked. The student is expected to organize their own learning independently. No prerequisites for the exam.

References

[Abh01] Shreeram Shankar Abhyankar. *Local analytic geometry.* Singapore: World Scientific, 2001.

[AM16] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Series in Mathematics. Westview Press, Boulder, CO, 2016.

[Fis76] Gerd Fischer. *Complex analytic geometry*, volume 538 of *Lect. Notes Math.* Springer, Cham, 1976.

[GR58] Hans Grauert and Reinhold Remmert. Komplexe Räume. *Math. Ann.*, 136:245–318, 1958.

[GR71] Hans Grauert and Reinhold Remmert. *Analytische Stellenalgebren. Unter Mitarbeit von O. Riemenschneider. (Analytic place algebras)*, volume 176 of *Grundlehren Math. Wiss.* Springer, Cham, 1971.

[GR77] Hans Grauert and Reinhold Remmert. *Theorie der Steinschen Räume*, volume 227 of *Grundlehren Math. Wiss.* Springer, Cham, 1977.

[GR84] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1984.

[Łoj91] Stanisław Łojasiewicz. *Introduction to complex analytic geometry. Transl. from the Polish by Maciej Klimek.* Basel etc.: Birkhäuser Veralg, 1991.

[MM10] Jeffery McNeal and Mircea Mustaţă, editors. *Analytic and algebraic geometry*, volume 17 of *IAS/Park City Mathematics Series.* American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2010.

[Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Annales de l'Institut Fourier*, 6:1–42, 1956.