Question 1 *Find a function with a local maximum in $x = 1$ and a local minimum in $x = 2$.*

Solution: Since we shall only find an example, let us make some simplifying assumptions. We need to control the sign of f'' in $x = 1$ and $x = 2$ to determine that we get a maximum/minimum at the appropriate points. Since we have two different types of extrema, a function with a simple sign change suffices. Thus let us make the ansatz

$$f''(x) = mx + n$$

and let us assume that $f''(1) = -1$ and $f''(2) = 1$. That ensures the correct signs for the extrema. From this condition, we obtain

$$m = \frac{f''(2) - f''(1)}{1} = 2$$

and with $f''(1) = 1$, we get

$$f''(1) = 2 + n = -1 \quad \Rightarrow \quad n = -3$$

and, thus,

$$f''(x) = 2x - 3. \tag{1}$$

Since f'' is a linear function, f' has to be a quadratic. Hence, we can make the ansatz

$$f'(x) = a(x-1)(x-2) = ax^2 - 3ax + 2a \tag{2}$$

since we know that the zeros of f' must be in $x = 1$ and $x = 2$. Now, we derive (2) and get

$$f''(x) = 2ax - 3a.$$

Comparison of coefficients with (1) leads to

$$a = 1.$$

Finally, since f' is a quadratic, f should be a cubic. Thus, we make the ansatz

$$f(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

and obtain

$$f'(x) = 3\alpha x^2 + 2\beta x + \gamma.$$

Thus, comparing the coefficients with (2) again, we obtain

$$\alpha = \frac{1}{3}, \quad \beta = -\frac{3}{2}, \quad \gamma = 2, \quad \delta \in \mathbb{R},$$

where δ is a free parameter which is not determined by the conditions of the question.[1] Putting all together, an example of an f satisfying the conditions of the question, would be

$$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x.$$

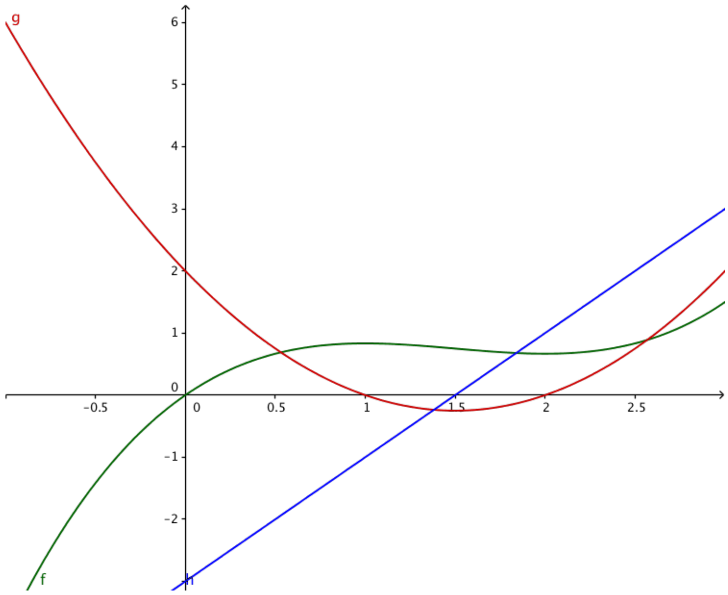

Figure 1: The graph of the function f (green), f' (red), and f'' (blue).

Question 2 (For the more experienced) *Could we additionally prescribe $f(1)$ and $f(2)$, e.g. $f(1) = 1$ and $f(2) = -1$.*[2]

Solution: A possible solution is

$$f(x) = 4x^3 - 18x^2 + 24x - 9.$$

[1] Why? What further condition could be asked?

[2] Hint: Yes. That we have more than one parameter free is hidden in the assumptions I made. Can you find it?

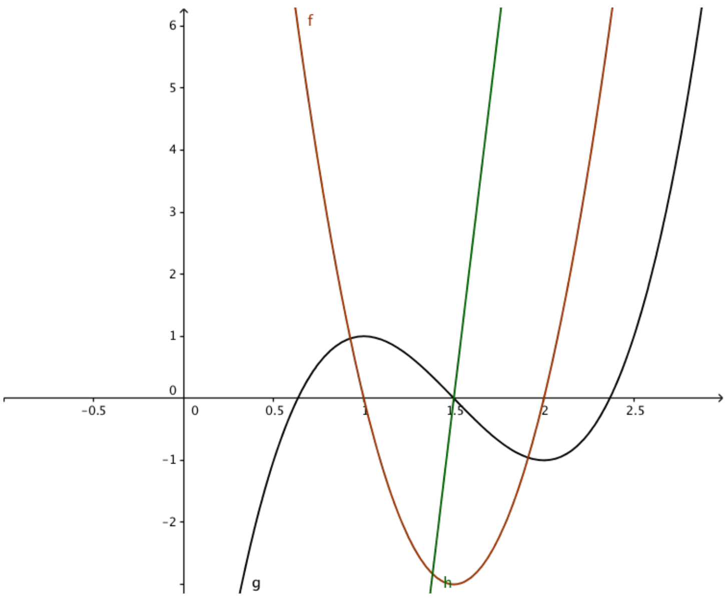

Figure 2: The graph of the function f and its 1st and 2nd derivative.

Question 3 *Design a question in the spirit of Question 1 that one could ask in an exam for this course. Give it to a peer and evaluate whether they understand the question and do what you like them to do.*