We want to find th stationary points of

$$f(x,y) = (x+3)y^2 + x^3 + 3x^2 - 45x.$$

First, we compute the first partial derivatives

$$\begin{aligned}\partial_x f(x,y) &= y^2 + 3x^2 + 6x - 45,\\ \partial_y f(x,y) &= 2(x+3)y.\end{aligned}$$

Now, from $\nabla f(x,y) = 0$, we obtain

$$\begin{cases} y^2 + 3x^2 + 6x - 45 = 0, \\ 2(x+3)y = 0. \end{cases} \tag{1}$$

We solve the second equation of (1) first: $2(x+3)y = 0$ is a product of real numbers and that means that it is zero if at least one of the factors is equal to zero. Thus, we get $x = -3$ and $y = 0$. Now we have to plug these values into the first equation of (1): the value $y = 0$ gives then

$$3x^2 + 6x - 45 = 0.$$

Now, we solve this equation with the pq-formula:

$$\begin{aligned}&3x^2 + 6x - 45 = 0 \quad \Leftrightarrow\\ &x^2 + 2x - 15 = 0.\end{aligned}$$

Thus, we get

$$x_{\pm} = -\frac{2}{2} \pm \sqrt{\frac{2^2}{4} - (-15)} = -1 \pm 4.$$

Thus, we get the stationary points $P_1(3,0)$ and $P_2(-5,0)$. The value $x = -3$ gives

$$y^2 + 3(-3)^2 + 6(-3) - 45 = y^2 - 36 = 0.$$

Thus, $y = \pm\sqrt{36} = \pm 6$. This gives us the stationary points $P_3(-3,6)$ and $P_4(-3,-6)$.

In the last test, many have determined the nature of the stationary points though that was not in the question. If the question asks for the stationary points only, we are done at this point.

Let us now suppose that the question asks also for the nature of the stationary points: First, we compute the second partial derivatives needed:

$$\begin{aligned}\partial_x^2 f(x,y) &= 6x + 6 = 6(x+1),\\ \partial_y^2 f(x,y) &= 2(x+3),\\ \partial_{xy}^2 f(x,y) &= 2y.\end{aligned}$$

We then compute $D(x, y)$:

$$\begin{aligned} D(x,y) &= \partial_x^2 f(x,y)\partial_y^2 f(x,y) - \left(\partial_{xy}^2 f(x,y)\right)^2 \\ &= 12(x+1)(x+3) - 4y^2. \end{aligned}$$

We now consider the points separately: $D(P_1) = D(3,0) > 0$. Since D is positive, we have to look at the sign of $\partial_{xx}^2 f(x,y)$. We get $\partial_{xx}^2 f(x,y) = 6(3+1) > 0$. This implies that P_1 is a local minimum. Now, $D(P_2) = D(-5,0) > 0$. Thus, we look at $\partial_x^2 f(-5,0) = 6(-5+1) < 0$. Thus P_2 is a local maximum. For P_3, we obtain $D(P_3) = D(-3,6) < 0$ which implies that P_3 is a saddle point. Finally, for P_4, we obtain $D(P_4) = D(-3,-6) < 0$ which implies that P_4 is a saddle point too.