Analysis I

Real numbers, sequences, and infinite series

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Foreword

These note have been written primarily for you, the student. I have tried to make it easy to read and easy to follow.

However, I do not wish to imply, that you will be able to read this text as it were a novel. If you wish to derive any benefit from it, read each page slowly and carefully. You must have a pencil and plenty of paper beside you so that you yourself can reproduce each step and equation in an argument. When I say *verify a statement*, *make a substitution*, etc. pp., you yourself must actually perform these operations. If you carry out the explicit and detailed instructions I have given you in remarks, the text, and proofs, I can almost guarantee that you will, with relative ease, reach the conclusions.

These wise words are borrowed from Morris Tenenbaum and Harry Pollard from the beginning of their book *Ordinary differential equations*. I could not have said it better and it certainly applies to this course.
The material is mainly taken from the Lecture Notes on Analysis I from Dr Lara Alcock and modified where I saw fit.

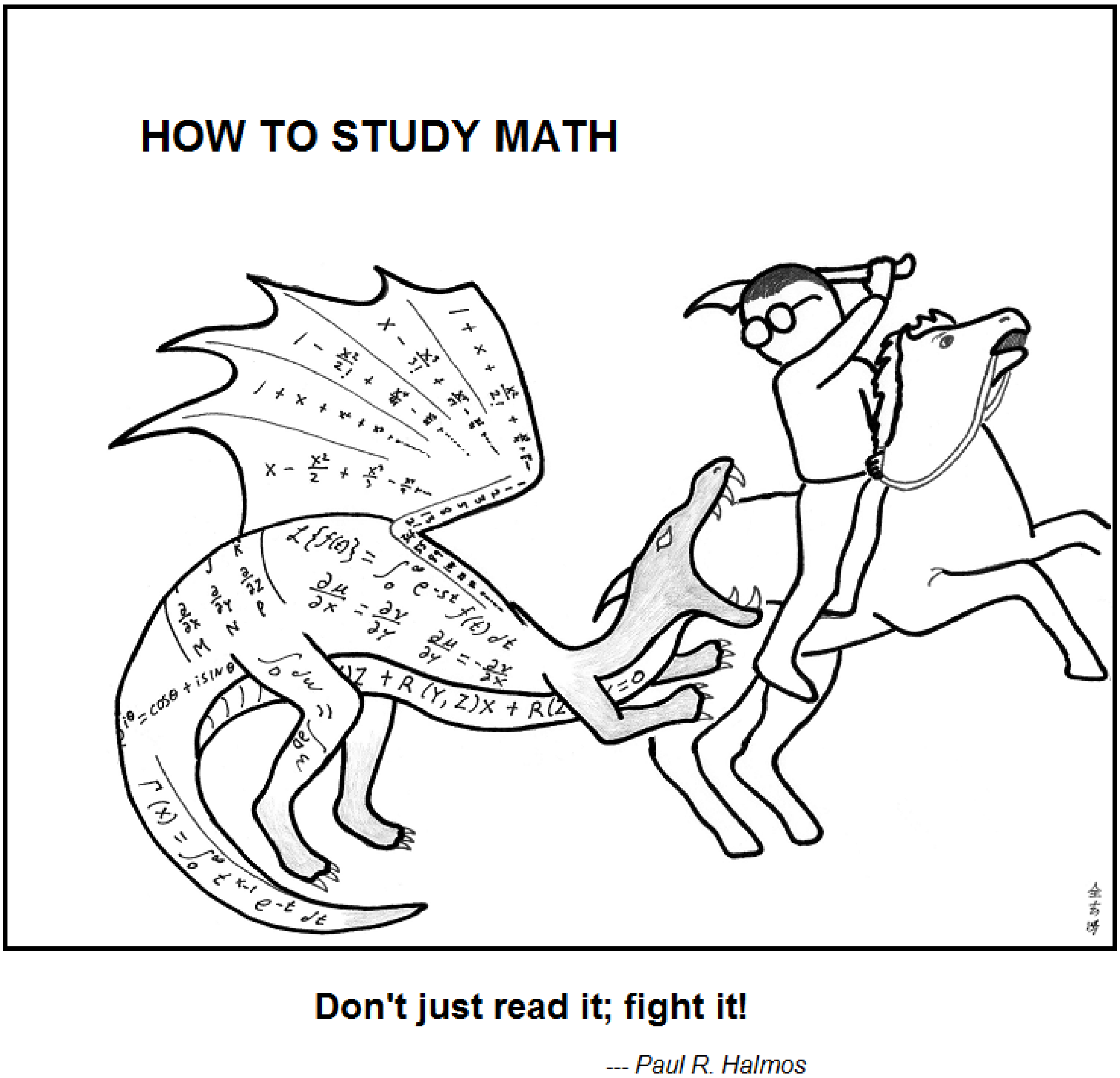

Figure 0.1: Don't just read it; fight it. – Paul Halmos (The comic is abstrusegoose.com)

Contents

Chapter

-1 What is Analysis?

This section is freely translated and adapted from Chapter 1.1. in [2]. I hold no intellectual properties of it as it is a slightly modified translation of the original German text. The pictures used and the founding documents of Analysis cited are in the public domain.

The 6th edition of Meyers Konversations-Lexikon lists under the keyword 'Analysis' the following entry:

> **Analysis** (Greek), a procedure of geometry (geometric A.) whose invention is generally attributed to Plato and which constitutes the opposite to the synthesis. As the latter takes the given and known and deduces the unknown and sought, the A. takes the sought as given and differentiates it and investigates the conditions under which it will hold until all relations to the known are found upon which the synthesis can go the other way.

Webster's Dictionary from 1913 says

> **Analysis** (Greek), A resolution of anything, whether an object of the senses or of the intellect, into its constituent or original elements; an examination of the component parts of a subject, each separately, as the words which compose a sentence, the tones of a tune, or the simple propositions which enter into an argument. It is opposed to synthesis.

These are interesting descriptions but not really helpful as it only says how Analysis operates but not to what it is applied. For Plato, this was clear. He saw Analysis as a method[1] of geometry. Meyer's encyclopaedia saw it it differently: 'Under Analysis one also understand all of mathematics with the exception geometry.' For the modern day mathematician, this definition is of course completely useless as one does of course not think of Algebra[2] as being part of Analysis and on the other hand quite some portions of Geometry count nowadays as Analysis. (One could conversely also say that big junks of Analysis belong to Geometry.) Thus it is still unclear what Analysis think about. Even the description of the method is questionable. As the encyclopedia notes: 'All clauses that speak a new truth, are also synthetic. Since the contents of most notions is not forever fixed but somewhat fluid is the same judgement an analytical one for some, a synthetic one for others'.[3] So in short, what is Analysis for one may be Synthesis for another and vice versa. In fact, the methods of modern Analysis, as in mathematics in general, are here analytic there synthetic and most of the time one does not indicate whether one does argue analytic or synthetic.

So far, we are not any closer to define the term 'Analysis'. At least we understood, that the term has changed its meaning with the flow of time and that we should understand it in historical context. Thus, let us look at the term 'Analysis' in the mathematical modern era (Neuzeit).

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[1]Greek ($\mu\varepsilon\theta o\delta o\zeta$): pursuit of knowledge, investigation, mode of persuing such inquiry. From $\mu\varepsilon\tau\alpha$ (after) and $o\delta o\zeta$(way, motion, journey).

[2]Meaning the mathematical field not what you know from school which is manipulation of formulas.

[3]You may consult the article *The Analytic/Synthetic Distinction* at the Stanford Encyclopaedia of Philosophy here.

Figure -1.1: The Swiss mathematician Leonhard Euler (1707–1783).

For instance, the Swiss mathematician Leonhard Euler (1707–1783), the most famous mathematician ot the 18. century, published 1748 a text book under the title *Introductio in analysin infitorum* what one could freely translate to *Introduction into the Analysis of the Infinite*. The table of contents (English here) makes it clear that it is a preparatory course for the differential and integral calculus. The author discusses the terms functions, series, chain fractions, and investigated the elementary functions as polynomials, rational functions, sine, cosine, logarithms, and exponentials. The second half of the book is devoted to curves and areas. The 'infinite' appears in the form of infinite series

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

and infinite chain fractions

$$[b_0,\, b_1,\, b_2,\, \ldots] = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \ldots}}}.$$

Some time later, Euler wrote a text book on Differential Calculus (Institutiones calculi differentialis, St. Petersburg 1755) and on Integral Calculus (Institutionum calculi integralis, St. Petersburg 1768-1770). These text books were an enormous influence on the following mathematicians, even the divide into the two branches of differential- and integral-calculus was kept for a very long time. Richard Courant (1888–1972) was the first who has discussed the Differential- and Integral-Calculus together in his still very readable text book Vorlesungen über die Differential- und Integralrechnung (Lectures on Differential- and Integral-Calculus) where he reached a fantastic clarity of the presentation of the Infinitesimal Calculus.

In 1797, after Euler's extensive treatises, a comparatively small book Théorie des fonctions analytiques, where the word 'analytic' is again in the title. This book had grown out of lectures that Joseph-Louis Lagrange (1736-1813) had given at the Ecole normale and Ecole polytecnique the great scientific institutions from the French Revolution era which done and continue to do enormously important contributions to mathematics, physics and engineering science.

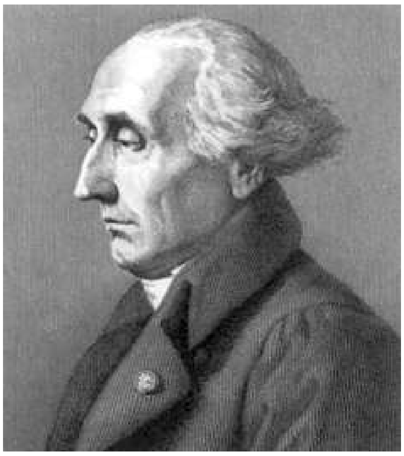


Figure -1.2: The Italian born French mathematician Joseph-Louis Lagrange (1736-1813).

We summarise: Under 'Analysis', we understand the field of the Differential- and Integral-Calculus together with their applications which the reader has already met in school. Of course, the material there is reduced to the most basic facts. Here, we will develop Analysis, which is besides Algebra and Geometry the main field of mathematics, so far that the reader will be able to follow the higher lectures of their programs. Lagrange's book has the sub-title *contentant les principes du calcul différentiel, dégagés de tout consideration d'infiniment petit, d'évannuissans, de limites et de fluxions, et reduits à l'analyse algébrique des quantités finies*. He announces that the discusses the main theorems of the differential calculus with algebraic analysis of finite quantities, freeing the considerations from the consideration of infinitely small quantities (which were introduced by Leibniz), from vanishing quantities (as Euler), from limits and from Newton's fluxions (another word for 'velocities' with which certain quantities change). In the first century after the discovery of the differential and integral calculus by Isaac Newton (1642-1726/27) around 1665 and Gottfried Wilhelm Leibniz (1646-1716) around 1672, the 'calculus' was developed with rapid speed without paying much attention to its foundations. Not only Lagrange has had doubts that the results found in that fashion would stand on solid ground. Already Bishop Berkeley has 1734 discussed this is an small text titled *The Analyst or a discourse addressed to an infidel mathematician*. The quite baroque title continues with *wherein it is examined whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidenlty deduced, than religious mysteries and poits of faith.*

Figure -1.3: The English mathematician Isaac Newton (1642-1726/27) and the German mathematician Gottfried Wilhelm Leibniz (1646-1716).

The bishop, a renowned philosopher, has angry with some of his quite freely thinking contemporaries which were very proud about the exact modern sciences and mocked religion as a ferry tale and yet their very foundations were such flimsy arguments as Newton's fluxions.
From Berkeley's book we can see that the contemporaries of Newton did understand nothing other than differential and integral calculus under the term 'Analysis'.

The mathematical modern times in Analysis start with the Prague religion philosopher and mathematician Bernard Bolzano (1781–1848) and the French mathematician Augustin-Louis Cauchy (1789–1857) who have introduce the notion of continuity into Analysis. Cauchy has in his lectures at Ecole polytechnique delivered a carefully crafted, strictly deductive foundation of Analysis. The two text books *Cours d'Analyse* (1821) and *Résum é des lecons donnés sur le calcul infinitesimal* (1823) were exemplary and had enormous influence on the following generations of mathematicians. Many French mathematicians have, following Cauchy's example, have published their lectures under the title *Cours d'Analyse*; one worth mention was the *Cours* from Camille Jordan (1838–1922).

Figure -1.4: The French mathematician Augustin-Louis Cauchy (1789–1857) and the German mathematician Bernard Bolzano (1781–1848).

The conclusion of her foundations has the Analysis found in Germany. Mainly in the lectures and works of Karl Weierstraß (1815–1897), Richard Dedekind (183–1916), and

Georg Cantor (1845–1918). Weierstraß taught Analysis in his Berlin lectures in proverbial rigour upon which nothing can be improved even today. The reader will encounter many of the definitions and results of Weierstraß,. It is interesting to note that the Weierstraß Lectures have never been published even though many people came to hear them from everywhere in Germany and from abroad. If at a given point there was no introductory lectures in Analysis, the students copied the notes of the older students and worked through those copies.

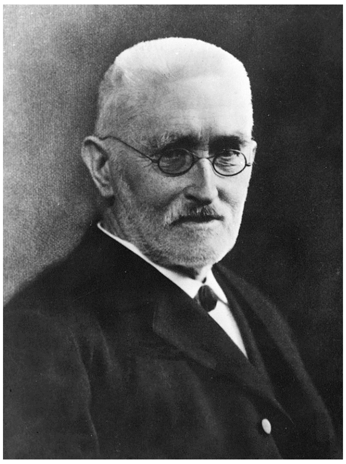
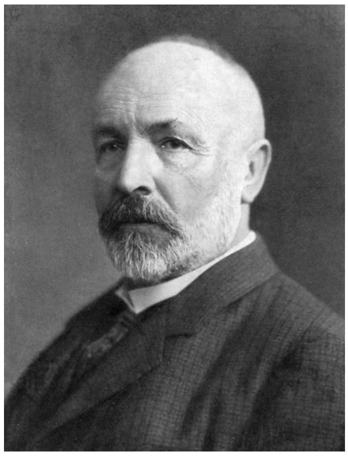




Figure -1.5: The German mathematicians Karl Weierstraß (1815–1897), Richard Dedekind (1831–1916), and Georg Cantor (1845–1918).

The apex in the erection of the cathedral of Analysis was the theory of the real numbers, especially the strict definition of irrational numbers by Cantor and Dedekind. We will give a short introduction into their work in the historical remarks on the real numbers in Section 1.1.

Chapter

0 Prerequisites

0.1 Some notation used in this notes

Symbols handwritten vs. typed. I tend to use the the hash (#) do indicate the end of a proof when I am writing something by hand. In this notes, the end of a proof will usually be indicated by a $\square$. If there is a short "proof" in a remark, no indication of its end will be given as it is understood that it should be clear. In these notes, I will not use any symbol to indicate contradictions in proofs by contradiction but simply state that we have reached one. In the notes that I make by hand in the lecture, I may use use a lightening bolt or just state the fact.

Notation in general. Please be aware that we expect yu to learn all notation presented in the lectures as soon as possible since we will use them all the time and you will get lost if you do not have a firm grasp on notation. For example, you should correctly use, $\in$ as element of and, as we will discuss in Section 2, clearly distinguish between a_n and (a_n). If you do not, you will definitively loose points in the class tests and the final exam.

The Greek alphabet. I assume that everyone is familiar with the Greek alphabet and knows how to read (meaning pronounce) and to write the letters. If you have problems with that, please do practice. Here is the table:

α	alpha	θ	theta	o	omikron	τ	tau
β	beta	ϑ	theta	π	pi	υ	upsilon
γ	gamma	γ	gamma	ϖ	pi	ϕ	phi
δ	delta	κ	kappa	ρ	rho	φ	phi
ϵ	epsilon	λ	lambda	ϱ	rho	χ	chi
ε	epsilon	μ	mu	σ	sigma	ψ	psi
ζ	zeta	ν	nu	ς	sigma	ω	omega
η	eta	ξ	xi				
Γ	Gamma	Λ	Lambda	Σ	Sigma	Ψ	Psi
Δ	Delta	Ξ	Xi	Υ	Upsilon	Ω	Omega
Θ	Theta	Π	Pi	Φ	Phi		

Table 0.1: Greek Letters

0.2 Some logic

We introduce the use of terms as *and* and *or* as they will be used throughout this text in statements of theorems.

In the following, the term proposition will mean a bearer of truth value, meaning it to be a statement that has either the value TRUE or the value FALSE. For instance, *Natural numbers are divisible by* 2*.* is a proposition bearing the truth value FALSE.

0.2.1 The conjunction and

In the English grammar, *and* is a conjunction that connects two words, phrases, or clauses. In mathematics (logic), the meaning is similar. We have two propositions, i.e. logical statements, say P and Q, that can be TRUE or FALSE. Then, the conjunction P and Q, in symbols $P \wedge Q$, is TRUE or FALSE depending on the truth values of P, and Q. In accordance with everyday meaning, $P \wedge Q$ is true if both, P and Q, are TRUE and is FALSE in any other case. We can write this also in a truth table:

P	Q	$P \wedge Q$
$TRUE$	$TRUE$	$TRUE$
$TRUE$	$FALSE$	$FALSE$
$FALSE$	$TRUE$	$FALSE$
$FALSE$	$FALSE$	$FALSE$

.

0.2.2 The conjunction or

In the English grammar, *or* is a conjunction that connects two words, phrases, or clauses. In mathematics (logic), the meaning is similar but there is a subtle difference to everyday usage.

For example, if you ask *Tea or coffee?* you mean of course *either* not that one can have both. However, in mathematics (logic) this is not the case. The conjunction or does always include the case that both propositions are TRUE. If we have two propositions, i.e. logical statements, say P and Q, that can be TRUE or FALSE, then, the conjunction P or Q, in symbols $P \vee Q$, is always true unless both propositions, P and Q, are FALSE. The truth table looks as:

P	Q	$P \vee Q$
$TRUE$	$TRUE$	$TRUE$
$TRUE$	$FALSE$	$TRUE$
$FALSE$	$TRUE$	$TRUE$
$FALSE$	$FALSE$	$FALSE$

.

0.2.3 Negation

In the English grammar, it is the process that turns an affirmative statement into its opposite denial. In logic, that is precisely the same. If P is a proposition which is TRUE or FALSE, then the proposition *not* P, in symbols $\neg P$, is FALSE or TRUE respectively, i.e.

P	$\neg P$
$TRUE$	$FALSE$
$FALSE$	$TRUE$

We have that $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ and $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$, where $\Leftrightarrow$ means that the left and the right hand side are logical equivalent, i.e. they have the same truth table. See also 0.2.6.

Exercise 0.1. *Prove the statements* $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ *and* $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ *by calculating the truth tables of both left and both right hand sides.*

0.2.4 Quantifiers

Oxford English Dictionary:

- (Logic) An expression (e.g. all, some) that indicates the scope of a term to which it is attached.
  - All humans are mortal.
  - All prime numbers are whole numbers.
- (Grammar) A determiner or pronoun indicative of quantity (e.g. all, both). Examples:
  - Every human will die.
  - Both, humans and animals are mortal.

Universal quantification

A universal quantification is a type of quantifier. It determines the scope of a logical constant as *for all*. We use the symbol $\forall$ in formulas.
Let us consider an example:

> The number 2 is not a divisor for any prime larger or equal to 3.

In symbols, we can write

$$\forall p \in \mathbb{P},\ p \geq 3\ :\ 2 \nmid p,$$

where we define $\mathbb{P} \subseteq \mathbb{N}$ as the set of all prime numbers. The symbol $\nmid$ means that the number on the left does not divide the number on the right. The symbol $\mid$ will be used to say the opposite, i.e. $n \mid m$ means that n divides m.

You may find further information in the Wikipedia article on the matter or in [3, 1].

Negated universal quantifiers are existential quantifiers. See also Section 0.2.5. For instance. *Every human will die* is negated as *There exists at least one human who is immortal*. See also the next section.

Existential quantification

An existential quantification determines that *there is at least one* object of some kind such that a logical proposition is true. For *there exists* or *there is at least one* (which are logically the same), we use the symbol $\exists$.
Let us consider an example:

> There exists a natural number greater than 2.

In symbols, we can write

$$\exists n \in \mathbb{N} \ : \ \ n > 2.$$

You may find further information in the Wikipedia article on the matter or in [3, 1].

Negated existential quantifiers are universal quantifiers. See also Section 0.2.5. For example, the above sentence *There exists a natural number greater than* 2., the negation is *All natural numbers are smaller or equal to* 2. Consider also the example at the end of the last section.

0.2.5 Negation of quantifiers

The negation of a statement is its logical negation; see also Section 0.2.3. There is a difference to everyday usage of the term. For example, as we will learn later, the negation of *the function* f *is increasing* is not *the function* f *is increasing* but *the function* f *is not increasing*. It is important that we care about what we say.

To get a feeling about how to negate statements, let us negate the two examples from universal and existential quantification. Before you read further, try it yourself and write negations down. Did you get them right?
The first statement was

> The number 2 is not a divisor for any prime larger or equal to 3.

Its negation would be

> There exists a prime number $p \geq 3$ such that 2 divides p.

In symbols

$$\exists p \in \mathbb{N},\ p \geq 3\ :\ 2 \mid p.$$

The second statement was

> There exists a natural number greater than 2.

Its negation is

> All natural numbers are smaller or equal than 2.

In symbols

$$\forall n \in \mathbb{N}\ :\ n \leq 2.$$

0.2.6 Implications and equivalences

If we want to say that an assumption P (a logical proposition) implies a conclusion Q (which is also a logical proposition), we can write $P \Rightarrow Q$. Please note that the arrow $\Rightarrow$ is not the same as $\rightarrow$ as the latter will take a different meaning later on.

If $P \Rightarrow Q$, we say that P is *sufficient* for Q and that Q is *necessary* for P.

The truth table is given by

P	Q	$P \Rightarrow Q$
$TRUE$	$TRUE$	$TRUE$
$TRUE$	$FALSE$	$FALSE$
$FALSE$	$TRUE$	$TRUE$
$FALSE$	$FALSE$	$TRUE$

.

Read and understand the table carefully. The last two lines mean that one can conclude everything from wrong assumptions. A problem that many people presenting 'logical' arguments do not consider.

For example, we can prove any statement about the empty set: Let $x \in \emptyset$, then $2 \mid x$. Here, P is the statement $x \in \emptyset$ and Q is the statement $2 \mid x$. Since P is always FALSE, we have

Theorem 0.1. *All elements of $\emptyset$ are divisible by 2.*

For more details you may consult the Wikipedia article on the matter.

If we want to say that two propositions P and Q are equivalent, we write $P \Leftrightarrow Q$. Again, please try not to write $\leftrightarrow$, learn and practice the right notation.

If $P \Leftrightarrow Q$, then we say that P is sufficient for Q and that Q is necessary for P and vice versa; we may also say that P is necessary and sufficient for Q and vice versa.

The truth table of is given by

$$\begin{array}{c|c|c} P & Q & P \Leftrightarrow Q \\ \hline TRUE & TRUE & TRUE \\ TRUE & FALSE & FALSE \\ FALSE & TRUE & FALSE \\ FALSE & FALSE & TRUE \end{array}. \tag{0.2.1}$$

Many people have problems distinguishing a proof by contraposition to a proof by contradiction.

A proof by contraposition is grounded on the fact that $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are equivalent, i.e.

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P).$$

That means that $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ have the same truth table. Let us prove that by supplying the second truth table:

$$\begin{array}{c|c|c|c|c} P & Q & \neg P & \neg Q & \neg Q \Rightarrow \neg P \\ \hline TRUE & TRUE & FALSE & FALSE & TRUE \\ TRUE & FALSE & FALSE & TRUE & FALSE \\ FALSE & TRUE & TRUE & FASLE & TRUE \\ FALSE & FALSE & TRUE & TRUE & TRUE \end{array}.$$

Compare that to Table 0.2.1.[1] Thus, if we prove something by contraposition, no contradictions are involved. We replace the statement that we want to prove by a logically equivalent one and prove that instead.

If we want to prove $P \Rightarrow Q$ by contradiction, we would assume that $P \wedge (\neg Q)$ is true and derive a contradiction which then by *tertium non datur*[2] implies statement $P \Rightarrow Q$ as Q must then be TRUE.
We will discuss this further, when we encounter proofs of this type in the lecture notes but I will make a couple of remarks.

0.2.7 Converses of statements

The converse of a statement is the logical negation of it. For example, let *All humans are mortal.* be our statement. Then, the converse is *There exists one human who is immortal.*. Be aware that the converse is *not All humans are immortal.*. Sometimes people use such constructions in everyday language. Look also back to Section 0.2.5.

0.2.8 Inverses of statements

An inverse of an statement is not to be confused with it negation or its contrapositive. Let us assume that $P \Rightarrow Q$ is our statement. Then, its inverse is $\neg P \Rightarrow \neg Q$.
For example, if the statement is $x < 10 \Rightarrow x < 11$, then the inverse is the statement $x \geq 10 \Rightarrow x \geq 11$. As you can see, if the statement is true, the inverse does not have to be true. Again, clearly distinguish inverse and converse!
However, one can prove that the inverse is the contrapositive of the converse and, hence, by our discussion in Section 0.2.6, the inverse is logically equivalent to the converse.

Example 0.1. *Consider the following implication: Let $n \in \mathbb{N}$.*

> *If n is a prime and $n \geq 3$, then it is not divisible by 2.*

The converse of the statement is

[1] Here is a good point where you should take pencil and paper and do all the tables again yourself without peaking in the notes. Can you do it?

[2] Means: no third exists. Also: Law of the excluded middle.

> *If n is not divisible by 2, then it is prime and $n \geq 3$.*

The contrapositive statement is

> *If n is divisible by 2, then it is not prime or $n < 3$.*

Finally, the inverse statement is

> *If n is not a prime or $n < 3$, then it is divisible by 2.*

Exercise 0.2. *Formulate the statements above by saying what is P and what is Q and then associate the following formal statements to the above sentences: $P \Rightarrow Q$, $\neg P \Rightarrow \neg Q$, $Q \Rightarrow P$, $\neg P \Rightarrow \neg Q$.*

Exercise 0.3. *Clarify for yourself that the inverse statement is logically equivalent to the converse statement in Exercise 0.1. Maybe calculate the truth table.*

0.3 Sets and operations on sets

Sets are collections of elements described by some property P. The standard notation for sets is

$$A = \Big\{x : x \text{ has property } P\Big\},$$

where one reads: A consists of all x such that x has property P.

Example 0.2. *Let us consider a set of numbers defined by the property that they are divisible by 2:*

$$\{x \ : \ \textit{there exists a natural number } k \textit{ such that } x = 2k\}\,.$$

I will assume that you are familiar with the meaning of some symbols described below.

Symbol	Description
$\mathbb{R}$	real numbers
$\mathbb{Z}$	whole numbers, i.e. $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{N}$	natural numbers, i.e. $\{1, 2, 3, 4, \dots\}$
$\mathbb{N}_0$	natural numbers containing 0
$\mathbb{Q}$	rational numbers
$\mathbb{C}$	complex numbers (will be introduced in a later module)

Table 0.2: Notation of certain sets.

We have the following inclusions

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

An important class of sets are subsets of the real numbers, called intervals. An interval is a set of numbers characterized by their left and right *boundary*. For example

$$[a,b] := \Big\{x \in \mathbb{R} : \quad a \leq x \leq b\Big\}$$

which we read as the closed interval a, b. Closed means that it contains a and b. An open interval does not contain the boundary points, i.e.

$$(a,b) := \Big\{x \in \mathbb{R} : \quad a < x < b\Big\}.$$

One can also consider the half-open cases

$$[a,b) := \Big\{x \in \mathbb{R} : \quad a \leq x < b\Big\}$$

and

$$(a,b] := \Big\{x \in \mathbb{R} : \quad a < x \leq b\Big\}.$$

We also denote $\mathbb{R} = (-\infty, +\infty)$. All numbers smaller than a would be denoted by $(-\infty, a)$, all numbers smaller or equal to a by $(-\infty, a]$. Similarly, one defines the sets of all numbers larger that or larger or equal to a given number. If we have the situation that we describe x as having either the property $x \geq a$ or $x \leq -a$ for a given $a \geq 0$, then we can write

$$\Big\{x \in \mathbb{R} : \quad x \geq a \quad \text{or} \quad x \leq -a\Big\}$$

which is the same as

$$x \in (-\infty, -a] \cup [a, +\infty).$$

Remark 0.1. *Some people write $]a,b[$ for (a,b) or $]a,b]$ for $(a,b]$ etc. pp. In this notes and my handwritten notes in the lectures, I will stick to the notation introduced above.*

Definition 0.1 (Intersection/Union/Difference)**.**
We denote by $A \cap B$ the ***intersection*** *of A and B which means that $A \cap B$ contains elements that are in A as well as in B. By $A \cup B$, we denote the* ***union*** *of the two sets A and B which means that $A \cup B$ contains elements that are either in A or in B. With $A \setminus B$, we denote finally the* ***difference*** *of A and B that means that $A \setminus B$ contains all elements in A that are not in B.*

Remark 0.2. *Of course the intersection and union is not limited to a finite number. If one has a family of sets $\{A_i : i \in I\}$ indexed by a countable or uncountable set I one can consider the sets $\bigcap_{i\in I} A_i$ and $\bigcup_{i\in I} A_i$. For Example:*

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} [-n, n], \qquad \{0\} = \bigcap_{n\in\mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right].$$

Definition 0.2 (Subset $A \subseteq B$)**.**
Let A and B be sets. Then, we say that A is a sub-set of B, in symbols, $A \subseteq B$ iff

$$\forall\, x \in A \quad \Rightarrow \quad x \in B.$$

We write $A \subset B$ if we want to signal that A is a proper sub-set of B, i.e. there are elements in B that do not belong to A.

0.4 Notes on Mathematical Writing

You should get into *good habits* when writing mathematics. One is to write down what you are using to justify any new claim. This is important because it helps to ensure that you are not assuming things you should not and it shows your reader that you can build an argument carefully. For instance, which is a better proof that if $a > b > 0$ and $c > d > 0$ then $ac > bd$?

$ac > bc$
$bc > bd$
So $ac > bd$.

We know that $ac > bc$ because $a > b$ and $c > 0$ (by transitivity).
Also $bc > bd$ because $c > d$ and $b > 0$.
Hence $ac > bc > bd$, so $ac > bd$ (by transitivity).

You recognise good mathematical writing when you see it, so use it yourself. Do not make your reader work harder than necessary to understand what you want. It is clear that I will not guess your meaning from your writing in the exam. It will be taken at face value.

0.4.1 Writing theorems

Theorems have *premises* and a *conclusion*.
The premises are things we assume are true (they are sometimes referred to as *hypotheses* or *assumptions*); the conclusion is something that follows if these premises hold.
Theorems are often written in one of these forms:

> If *these premises* hold then *this conclusion* holds.
>
> For every object that satisfies *these premises*, *this conclusion* holds.
>
> Let *these premises* hold. Then *this conclusion* holds too.

An *if...then* statement is sometimes known as a *conditional statement*. Conditional statements can be written in different ways: One could just write $A \Rightarrow B$ or use words as in *If* A *then* B .... I suggest you stick to words as long as necessary. It is more important to say something correct than to say something short. However, you should be able to read both equally fluently. A good exercise is to take the theorems that we will discuss during the semester and write them in different forms.

Exercise 0.4. *Consider the statement* $x < 2 \Rightarrow x < 5$. *Formulate its converse, its inverse, and its contrapositive.*

At some point, you will be tempted to use the untrue converse of a true statement. That is not your fault in so far as the everyday language is sometimes lazy about the distinctions. However, you need to constantly work on your comprehension of the language. You need to learn the meaning of the terms introduced very quickly as we will use them all the time and you can not follow the discussion in class if you try to learn them just before the exam.

When you think about a theorem we discussed in class, you should ask yourself at home about the converse and why or why not it may hold.

Let us say that we have a theorem that has premises A, B, and C and says that then D holds true. First you should to try to find examples where D holds but at least one of the conditions A, B, or C does not hold. Then, the converse of the theorem can not hold.

Additionally, to understand the importance of the conditions, you should find examples that satisfy premises A and B but not C and also not D as well as examples which satisfy conditions A and C but not B and not D etc. pp. This will help you to understand the theorems and also might give insights in proofs.

Chapter

1 Real Numbers

The discussion of this chapter follows [2, Ch. 3] and some Lecture Notes taken from Analysis lectures given by Prof. Elias Wegert at TU Bergakademie Freiberg.

There are different methods to introduce the real numbers. Here, we will follow the axiomatic method. That means we do not define the objects themselves but describe the operations that can be done with them and which properties they have.
These rules are called axioms and will be assumed, i.e. postulated. All other propositions are derived from those axioms by logical deduction.

Advantage: Honesty. We clearly state which assumptions are used for a proof.
Disadvantage: Very formal, not very intuitive.

As our reasoning must stand of firm grounds, we will prefer honesty over intuition at the moment. I will try to add plenty of intuition where I can and you will get a better and clearer picture the more you learn.

So, what are the real numbers? The answer to this question leaves the mathematician, as I indicated above, to the philosophers. He does not ask *What are numbers?* but *How does one operate with numbers?*

Similar to a chess player who describes the pawns by defining how they may act, mathematicians describe the real numbers by the rules which describe how one may operate with the numbers. Again, those rules are called **Axioms** of the real numbers.

1.1 Historical remarks

At the foundation of Analysis are the real numbers. The notion of the real numbers was developed in a long historical process which has its beginnings in the grey fog of pre-history where we know almost nothing about how humans finally learned to count and will probably forever be concealed. The process of describing the real numbers axiomatically and constructing them self-consitently came to its end no sooner then the end of the 19th century in the work of Georg Cantor in his creation of set theory.
At the beginning stand the natural numbers

$$1,\ 1+1=2,\ 1+1+1=3,\ \ldots$$

which are learned by every child. They can be used for two goals. One is *counting*, for example apples in a bowl. The second one is to *order* given things by *numbering*, for instance the pages of the present Lecture Notes. In the first case, one uses the numbers as *cardinals*, in the second as *ordinals*. Most languages distinguish between those two functions of the natural numbers. In English for example, we have the cardinal numbers *one*, *two*, *three*, ...and ordinal numbers *first*, *second*, *third*, ... It is an old question of debate whether one of the notions is more fundamental or whether both are of equal value and should be seen as independent. Leopold Kronecker (1831–1916) has famously said that the natural numbers were created by god and all the rest of mathematics is human creation.

Figure 1.1: The German mathematician Leopold Kronecker (1831–1916).

Humans have learned to add natural numbers n, m and also to multiply them. The sum $m + n$ and the product $n \cdot m$ are again members of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers. However, if one subtracts arbitrary natural numbers, one does in general not get a natural number. The difference $n - m$ is only a natural number if n is larger than m. Also, one can not divide two arbitrary natural numbers p, q as the quotient $\dfrac{p}{q}$ is only in $\mathbb{N}$ if q is a divisor of p. That means that one can not solve algebraic equations as

$$m + x = n \tag{1.1.1}$$

or

$$q \cdot x = p \tag{1.1.2}$$

with $x \in \mathbb{N}$. To remedy the situation, the zero 0 and the negative numbers -1, -2, -3, ...were introduced and $\mathbb{N}$ was extended to the whole numbers $\mathbb{Z}$

$$\mathbb{Z} = \{\ldots - 2, -1, 0, 1, 2, \ldots\}.$$

The symbol $\mathbb{Z}$ comes from the German word *Zahl* for number. In $\mathbb{Z}$, one can solve equations of the type (1.1.1) with arbitrary given numbers $m, n \in \mathbb{Z}$. To solve equations of the type (1.1.2) for every $p, q \in \mathbb{N}$, we have to introduce another kind of numbers, the fractions $\frac{p}{q}$ which are also called positive rational numbers. It is worth noting, that fractions were historically introduced before the zero and the negative numbers. In the middle ages, negative numbers were still viewed as arcane or mystical numbers.

More general, one can consider the set of all quotients $\frac{p}{q}$ of whole numbers p, q with $q \neq 0$. This set is denoted by $\mathbb{Q}$ which comes from the German word (of Latin origin) Quotient for quotient. We have

$$\mathbb{Q} := \left\{ \frac{p}{q} \ : \quad p \in \mathbb{Z}, \ q \in \mathbb{N} \right\}.$$

In this set, one can add, subtract, multiply, and divide without any restrictions following the well known rules. Furthermore, one can solve linear algebraic equations of the form

$$ax + b = 0 \tag{1.1.3}$$

for arbitrary $a, b \in \mathbb{Q}$ with $a \neq 0$ uniquely by an $x \in \mathbb{Q}$.

A remarkable fact is that equation (1.1.3) has in general no solution in $\mathbb{N}$ and the solvability of (1.1.3) can be forced by adding some 'ideal' (i.e. imagined) elements. In your studies, you will see similar processes, comparable to the extension of $\mathbb{N}$ to $\mathbb{Z}$ and $\mathbb{Z}$ to $\mathbb{Q}$ when you will learn how we find numbers in which we can solve polynomial equations as $x^2 + 1 = 0$.

The Pythagoreans in ancient Greece believed that all relations of lengths that occur in nature can be expressed in rational numbers. Their word view crumbled as a member discovered that there are pairs of line segments that are incommensurable, i.e. their measured values are in non-rational relation. To understand what that means, we must interpret the fraction $\dfrac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ geometrically. We consider a straight line, the number line $\mathcal{G}$, which we orient by specifying the direction $\rightarrow$.

Figure 1.2: Number line $\mathcal{G}$.

Now, we fix a point on the line that we call 0 and now we tick a point to ´the right' which will define the unit length. The right end point of the unit line segment is called 1. Repeating this to the left and the right, we get the whole numbers $\mathbb{Z}$, where we but -1, -2, ...to 'the left' of 0. Now, the right endpoint of the qth part of the unit line segment, $q \in \mathbb{N}$, gets the label $\dfrac{1}{q}$ and the p fold of $\dfrac{1}{q}$, $p \in \mathbb{N}$, marked to the right, gets the label $\dfrac{p}{q}$ etc pp.

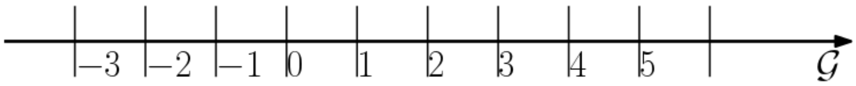


Figure 1.3: Number line $\mathcal{G}$ with ticks of natural numbers.

Exercise 1.1. *Draw the rational numbers in the above number line in a reasonable way. Think about how you actually could construct a number segment which is, say $\frac{1}{7}$th of the unit segment.*

Thus, the rational numbers are dispersed on the number line $\mathcal{G}$ like 'infinitely small' pearls on a chain. Now we think ourself a given line segment to the right of zero. If the line segment were commensurable, i.e. the length were in a rational relation to the unit line segment, then the right end of the segment would be a rational number. We will now see that this is not always the case. For that, we erect a unit square over the unit line segment. The diagonal of that square can be 'projected' on the number line with a compass by taking the diagonal as the radius and ticking the number line.

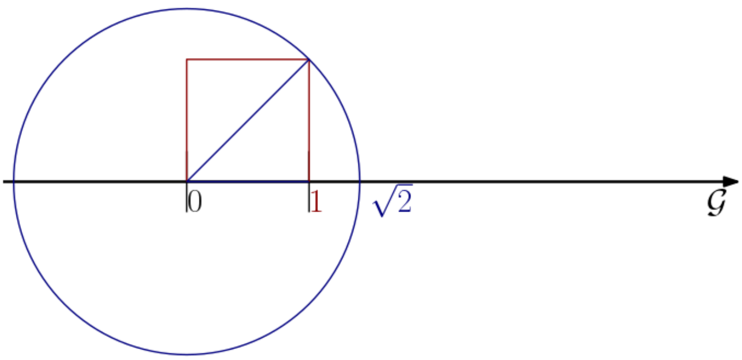


Figure 1.4: Number line $\mathcal{G}$ with ticks of natural numbers.

We will now prove, that the length of the resulting line segment is incommensurable. By the Pythagorean theorem, the diagonal has the length $\sqrt{2}$. If $\sqrt{2}$ were not irrational, we could write

$$\sqrt{2} = \frac{p}{q}$$

for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ and we can assume, without loss of generality, that p and q have no common divisors.[1] Squaring both sides of the equation and multiplying both sides with q^2, we obtain

$$2q^2 = p^2.$$

Thus, p^2 is divisible by 2 and, as 2 is prime, p is divisible by 2 as well. Hence, q^2 is divisible by 2 and therefore q is divisible by q. That means that p and q are both divisible by 2 which contradicts our assumption that $\gcd(p, q) = 1$.

[1] If a and b have no common divisor, we also say that they are co-prime. In symbols $(a, b) = 1$ or $\gcd(a, b) = 1$.

Exercise 1.2. *Use the Self-Explanation Method to study the proof above.*

Exercise 1.3. *Do the above proof again and show that* $\sqrt{3}$ *is not rational.*

Exercise 1.4. *Can you use the strategy to prove that $\sqrt{p}$ is irrational for any prime p? Is it necessary that p is a prime? Give good reasons for your statements.*

Exercise 1.5. *Is $\sqrt{6}$ irrational? If yes, does the above proof work? Can you come up with a rather general statement for which numbers $m \in \mathbb{N}$, we have that $\sqrt{m}$ is irrational? What you need to know is the* fundamental theorem of arithmetic, *also called the unique factorization theorem. See here.*

1.2 Axioms of the real numbers

In this section, we introduce the axioms of the real numbers. In total there will be 16 axioms which describe the real numbers. Going back to our analogy with the chess game, the reader should understand that 0.1123487 is not a real number per se but only a mere representation (the so-called decimal expansion) of a real number as any representation of a 'knight' is not 'the' knight. What a knight is is defined by the moves he is allowed to do.
The real numbers are an abstract set whose existence we suppose together with the well known operations addition and multiplication as well as their inverse operations. Now let us be more precise.

There exists a set $\mathbb{R}$ of elements a, b, c, .. which we call real numbers that fulfil the following three groups of axioms.

(I) The algebraic axioms.

(II) The ordering axioms.

(III) The completeness axiom.

We will now describe the Axioms (I)–(III) in detail.

The algebraic Axioms (I). There exist two operations on $\mathbb{R}$, called addition and multiplication, which assign to every pair a, b of elements from $\mathbb{R}$ two new elements $a+b \in \mathbb{R}$ and $ab \in \mathbb{R}$ (we set $ab = a \cdot b$). They are called the sum and the product of a and b. The operations addition and multiplication satisfy the following rules.

(I.1) $(a+b)+c = a+(b+c)$ (Associativity)

(I.2) $a+b = b+a$ (Commutativity)

(I.3) There is exactly one element in $\mathbb{R}$, called the zero and denoted by 0, such that $a+0 = a$ for all $a \in \mathbb{R}$.

(I.4) For all $a \in \mathbb{R}$ there exists exactly one $b \in \mathbb{R}$ such that $a+b = 0$. The element b will be denoted by $-a$ and we will call it the negative to a.

(I.5) $(ab)c = a(bc)$ (Associativity)

(I.6) $ab = ba$ (Commutativity)

(I.7) There is exactly one element in $\mathbb{R} \setminus \{0\}$, called the one and denoted by 1, such that $a \cdot 1 = a$ for all $a \in \mathbb{R}$.

(I.8) For every $a \in \mathbb{R} \setminus \{0\}$ there is exactly one element $b \in \mathbb{R}$ such that $ab = 1$. We denote b by a^{-1} or $\frac{1}{a}$ and say that a^{-1} is the inverse element to a.

(I.9) $a(b + c) = ab + ac$ (Distributivity)

Notation. We set

$$a - b := a + (-b), \quad \frac{a}{b} := ab^{-1} = b^{-1}a$$

and call $a - b$ the difference of a and b and $\dfrac{a}{b}$ the quotient of a and b. The operations $a, b \mapsto a - b$ respectively $\dfrac{a}{b}$ are called subtraction and division.

Remark 1.1. *In (I.3), we assume that the zero is unique. However, it would be enough to postulate that there exists an element $0 \in \mathbb{R}$ such that $a + 0 = a$ for all $a \in \mathbb{R}$. One can prove the uniqueness.*

Exercise 1.6. *Prove the assertion in Remark 1.1.*

Remark 1.2. *In (I.4), we assume that the negative element $-a$ to an element $a \in \mathbb{R}$ is unique. However, it would be enough to postulate that there exists such an element. One can prove the uniqueness.*

Exercise 1.7. *Show the uniqueness of the inverse element (see (I.8)).*

Exercise 1.8. *Similar statements as in Remarks 1.1 and 1.2 are true for 0 and a^{-1}. Formulate and prove them.*

Remark 1.3. *To prove the above assertions, we have introduced important tricks which could be named adding an active zero or multiplying an active one. Where and how? You are advised to remember these tricks very well.*

Remark 1.4. *If you look carefully at the rules, you can see that they come as four rules for addition, four rules for multiplication and one rule that shows how they interact. In further modules such as* Linear Algebra, Algebra and Numbers, *and* Geometry and Groups, *you will see other structures satisfying part of those or similar axioms. These play a fundamental role in modern mathematics as they help us to keep a common language and to order our fields.*

Remark 1.5. *Following Dedekind*[2]*, a set* $\mathbb{F}$ *whose elements have properties (I.1) to (I.9) is called a field and thus one can call* $\mathbb{R}$ *the field of real numbers. You will learn more about this in the module* Numbers*. There you will also learn how one can actually construct the real numbers. In that context one can prove the axioms that we take for granted here. The price we have to pay are some other axioms that we have to accept.*

Derived rules: Using (I.1) to (I.9), one can now prove many more rules that you might have taken for granted so far. The use of this exercise is to put everything we do on firm ground by making as few assumptions as possible and deriving as much from those as we can.

(I.10a) $-(-a) = a$ for all $a \in \mathbb{R}$

(I.10b) $(-a) + (-b) = -(a + b)$ for all $a, b \in \mathbb{R}$,

(I.10c) $(a^{-1})^{-1} = a$ for all $a \in \mathbb{R} \setminus \{0\}$,

(I.10d) $a^{-1}b^{-1} = (ab)^{-1}$ for all $a, b \in \mathbb{R} \setminus \{0\}$,

(I.10e) $a \cdot 0 = 0$,

(I.10f) $a(-b) = -(ab)$ for all $a, b \in \mathbb{R}$,

(I.10g) $(-a)(-b) = ab$ for all $a, b \in \mathbb{R}$, and

(I.10h) $a(b - c) = ab - ac$ for all $a, b, c \in \mathbb{R}$.

Further, we can show the very important rule

(I.11) If $ab = 0$ then at least one of the two numbers a, b is equal to zero.

Let us prove some of the (I.10x), $x \in \{a, \ldots h\}$. The rest will be on the problem sheet for you to sort out.

[2]Richard Dedekind (1831–1916), German mathematician.

Remark 1.6. *Let me stress that $-a$ is a symbol for the element $b \in \mathbb{R}$ such that $a + b = 0$. That $-a = (-1) \cdot a$ is something that we have to prove. In fact, we already did. It is a consequence of (I.10f) with $b = 1$.*

Proof of (I.10a). (Indicate the laws that we use.) We have

$$\begin{aligned} a = a + 0 &= \underbrace{a + (-a)}_{=0} + (-(-a)) \\ &= -(-a) \end{aligned}$$

□

Proof of (I.10b). (Indicate the laws that we use.) We have

$$\begin{aligned} 0 = a + (-a) &= a + (-a) + b + (-b) \\ &= a + b + (-a) + (-b) \\ &= (a + b) + (-a) + (-b) \end{aligned}$$

And thus, since the inverse element $-(a+b)$ to $a+b$ is unique, we have $(-a)+(-b) = -(a + b)$. □

Proof of (I.10c). (Indicate the laws that we use.) Again, we know $(a^{-1})^{-1} \cdot a^{-1} = 1$. Look at the proof of (I.10a) and produce one for (I.10c).

Proof of (I.10d). (Indicate the laws that we use.) We have $a \cdot 0 = a(0+0)$. From that we get

$$a \cdot 0 = a \cdot 0 + a \cdot 0 \quad \Rightarrow \quad 0 = a \cdot 0.$$

$\square$

Proof of (I.11). (Indicate the laws that we use.) Let $a \neq 0$ and $ab = 0$. Then, we have

$$\begin{aligned} b &= 1 \cdot b = (a^{-1}a)b \\ &= a^{-1} \cdot 0 = 0 \end{aligned}$$

$\square$

Finally, we have the following rules whose proof is on the Problem Sheet.

(I.12a) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad+bc}{bd}$, for all $a, b \in \mathbb{R}$ and $c, d \in \mathbb{R} \setminus \{0\}$,

(I.12b) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ for all $a, b \in \mathbb{R}$ and $c, d \in \mathbb{R} \setminus \{0\}$, and

(I.12c) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc}$ for all $a \in \mathbb{R}$ and $b, c, d \in \mathbb{R} \setminus \{0\}$.

Ordering axioms (II). For arbitrary real numbers a, b it is always clear whether they are equal ($a = b$) or unequal ($a \neq b$). We postulate that there is a relation , denoted by $<$ such that for two different, i.e. unequal a and b, exactly one of the two relations $a < b$, $b < a$ is true.

We call that the **trichotomy axiom**. For arbitrary $a, b \in \mathbb{R}$, hold exactly one of the three relations

$$a < b, \quad a = b, \quad b < c.$$

The ordering relation satisfies the following axioms for all $a, b \in \mathbb{R}$.

(II.1) From $a < b$ and $b < c$ follows $a < c$. (Transitivity)

(II.2) From $a < b$ follows $a + c < b + c$ for all $c \in \mathbb{R}$.

(II.3) From $a < b$ and $c > 0$ follows $ac < bc$.

One says that (II.2) is compatibility of the ordering relation with the addition and (II.3) is compatibility of the ordering relation with multiplication.

Notation and terminology. We read $a < b$ as a *is smaller than* b. We will also use the equivalent notation $b > a$ which we then read as b *is larger than* a. The relation $a \leq b$ says that either $a = b$ or $a < b$ ans e say that a *is smaller or equal to* b. Similarly, we define $a \geq b$. Further, we say that $a \in \mathbb{R}$ is positive if $a > 0$ and negative if $a < 0$. We say that a is non-negative or non-positive if $a \geq 0$ or $a \leq 0$ holds.

Derived rules. As before, we can drive further rules from (II.1) to (II.9), where we can of course also use all rules (I.1) to (I.12) and (II.1) to (II.3) insofar as we have proven them or they are axioms.

(II.4) $a < b \;\Leftrightarrow\; b - a > 0$,

(II.5) $a < 0 \;\Leftrightarrow\; -a > 0$ and $a > 0 \;\Leftrightarrow\; -a < 0$,

(II.6) $a < b \;\Leftrightarrow\; -b < -a$,

(II.7) $ab > 0 \quad \Leftrightarrow a > 0,\, b > 0$ or $a < 0, b < 0$,
$ab < 0 \quad \Leftrightarrow a > 0,\, b < 0$ or $a < 0, b > 0$,

(II.8) $a \neq 0 \Leftrightarrow a^2 > 0$ (in particular $1 > 0$),

(II.9) From $a < b$ and $c < 0$ follows $ac > bc$.

(II.10) $a > 0 \quad \Leftrightarrow \quad \dfrac{1}{a} > 0$,

(II.11) From $a^2 < b^2$, $a \geq 0$, and $b > 0$ follows $a < b$.

(II.12) If $a < b$ and $x < y$, then $a + x < b + y$.

Exercise 1.9. *Try to prove (II.4) to (II.12). Remember to use only what is already known to be true, either by axiom or we already proved it.*

Proof. Carefully read the proofs and add information you may need to remember how to go about this problems.

(II.4) We have that $-a \in \mathbb{R}$ and , since $a < b$, we get by (II.2) that

$$\underbrace{a + (-a)}_{=0} < b + (-a)$$

which proves $[\Rightarrow]$ of (II.4). What about the converse?

(II.5) The statement $a < 0$ is (II.2) with $b = 0$. Since $-a \in \mathbb{R}$ (I.4) with $(-a) + a = 0$, we get

$$\underbrace{a + (-a)}_{=0} < 0 + (-a) = -a$$

What about the converse? Now, if $a > 0$, we have $-a \in \mathbb{R}$ and, $a > 0$ is (II.2) with $a = 0$ and $b = a$. The rest follows as before. (The statement also follows directly from (II.4) with $b = 0$ and (I.10a).)

(II.6) Since $a < b$, we have, by (II.4) that $b - a > 0$. From (I.10a/b), we get

$$-((-a) + b) = a + (-b) = a - b.$$

Since $b - a = -a + b$, by (I.2), we then can use (II.5) to get

$$-(b - a) = a - b < 0.$$

Using (II.2) (How exactly?), we get $-b < -a$.

(II.7) For purposes of presentation, we prove both directions seperately. First, we prove $[\Rightarrow]$. We have that $a > 0$, $b > 0$ implies by (II.3) that $ab > 0$. Further, if $a < 0$, $b < 0$, we get by (II.5) that $-a > 0$ and $-b > 0$. Now, by (II.3) and (I.10g), we obtain $0 < (-a)(-b) = ab$. (Thus, if $a > 0$, $b > 0$ OR $a < 0$, $b < 0$ is true, then $ab > 0$ is true.)
Now we prove $[\Leftarrow]$. We assume that $ab > 0$ and, that $a > 0$, $b < 0$. We have $-b > 0$ and by (II.3) that $0 < a(-b) = -ab$. Thus, by (II.6), we get $ab < 0$ which contradicts our assumption. Obviously $a < 0$, $b > 0$ works

with the same argument (renaming). Therefore, we could make the assumption $a > 0$, $b < 0$ without restriction of generality. Thus, $a > 0$, $b > 0$ or $a < 0$, $b < 0$.

(II.8) Follows immediately from (II.7).

(II.9) From $a < b$, we get that $a - b < 0$. There we use (II.4) and (II.5). From $c < 0$, we get $-c > 0$ from (II.5). Using (II.3), we obtain $a(-c) < b(-c)$ which gives $-ac < -bc$ by (I.10f). By (II.6), we get $bc < ac$.

(II.10) If $a > 0$, we can write $a \cdot (a \cdot a^{-1}) = a^2a^{-1} > 0$. By (II.8), we have that $a^2 > 0$ and then by (II.7) that $a^{-1} > 0$. (The transformations are all equivalence transformations. Make that more explicit.)

(II.11) Note that $(-b)b = -b^2$ by (I.10f). Then we can show the identity

$$(b - a) \cdot (a + b) = b^2 - a^2.$$

By (II.4) we have $b^2 - a^2 > 0$ from the assumption $a^2 < b^2$, Thus, by (II.7), we get that $b - a > 0$. Using (II.2) by adding a to both sides, we finally obtain $a < b$.

$\square$

Exercise 1.10. *The proof of (II.12) is on the Problem Sheet for you to figure out.*

Remark 1.7. *Some of you might have been tempted to take the root in (II.11) to prove the result. However, that assumes some things we do not yet know. First, it assumes that the root keeps the direction if the $<$ relation (this is called monotonicity) and, secondly, it assumes that there is even such a thing. So far, we have no proof that for a given $y \in \mathbb{R}_{\geq 0}$ there is a $x \in \mathbb{R}$ such that $x^2 = y$.*

We are now ready to come to the final axiom which, as we will see in the sequel, makes Analysis possible. In a way, as I will detail later on, the completeness axiom makes sure that there are no holes in the number line.

The completeness Axiom (III). We formulate the completeness axiom in a way that essentially dates back to the work of Dedekind. There are other possibilities and I will discuss this later in Chapter 2.

Figure 1.5: Some thoughts on the number line.

To state the axiom, we need some more setup. Recall that a set S is said to be non-empty if it has at least one element. In symbols, we write $S \neq \emptyset$. Further, we introduce

Definition 1.1 (Types of boundedness)**.**
Let $A \subseteq \mathbb{R}$. Then,

1. *we say that A is **bounded above** iff there exists a constant $C \in \mathbb{R}$ such that*
$$\forall x \in A : \; x \leq C.$$
Such a C is called an upper bound of A.

2. *we say that A is **bounded below** iff there exists a constant $C \in \mathbb{R}$ such that*
$$\forall x \in A : \; x \geq C.$$
Such a C is called a lower bound of A.

3. *we say that A is **bounded** iff A is bounded above and bounded below.*

Exercise 1.11. *Clearly formulate what it means for a set to be unbounded. cm*

Exercise 1.12. *Give a definition of boundedness with an inequality.*

Definition 1.2 (Supremum of a set $A \subseteq \mathbb{R}$)**.**
*Let $A \subseteq \mathbb{R}$. Then $U \in \mathbb{R}$ is the **supremum** of A if and only if:*

1. *$\forall a \in A$, we have $a \leq U$;*
2. *if u is an upper bound for A, then $U \leq u$.*

We denote the supremum of A by $\sup(A)$.

The supremum is sometimes called the **least upper bound**. Can you see why?

Definition 1.3 (Infimum of a set $A \subseteq \mathbb{R}$)**.**
*Let $A \subseteq \mathbb{R}$. Then $L \in \mathbb{R}$ is the **infimum** of A if and only if:*

1.
2.

We denote the infimum of A by $\inf(A)$.

What do you think the infimum is sometimes called?

Exercise 1.13. *Can you justify the use of the definite article 'the' in both definitions? (That means can you justify that the infimum and supremum is unique if it exists? Otherwise, we should have used 'a'.)*

Exercise 1.14. *Let* A *be a non-empty set and assume that* $\sup(A)$ *and* $\inf(A)$ *exist. Prove that*

$$\inf(A) \leq \sup(A).$$

Set	Bnd. above?	3 upper bounds	Supremum
$[0, 1]$			
$(0, 1)$			
$[0, \infty)$			
$\left\{x \in \mathbb{R} \mid x^2 < 2\right\}$			
$\left\{x \in \mathbb{R} \mid x^2 \leq 2\right\}$			
$\left\{x \in \mathbb{Q} \mid x^2 \leq 2\right\}$			
$\{1 + 1/n \mid n \in \mathbb{N}\}$			
$\{1 + (-1)^n/n \mid n \in \mathbb{N}\}$			
$\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \mid n \in \mathbb{N}\right\}$			

Table 1.1: Some examples of upper bounds for some sets as well as suprema if they exist.

Exercise 1.15 (True or false?)**.**

- *If a set* $A \subseteq \mathbb{R}$ *has supremum* U*, then* $U \in A$.

- *If a set* $A \subseteq \mathbb{R}$ *has supremum* $\sup(A)$ *and if we define* $-A = \{-a|a \in A\}$*, then* $\sup(-A) = -\sup(A)$.

- *If a set* $A \subseteq \mathbb{R}$ *has supremum* $\sup(A)$. *Then for all* $\varepsilon > 0$ *there exists* $a \in A$ *such that* $\sup(A) - \varepsilon < a \leq \sup(A)$.

Reminder: A diagram is ***not*** *usually considered enough for a proof – we should translate any insight gained via diagrams into an argument based on definitions and/or on other accepted results. That said, Analysis lends itself to diagrammatic representations because we often consider sets or sequences of real numbers – which can be represented on number lines – or sequences and (later) other functions – which can be represented on graphs.*

Lemma 1.1.

Some thinking first.

Proof.

With that we are ready to formulate

(III) Every non-empty, bounded above sub-set of $\mathbb{R}$ has a smallest upper bound, i.e has a supremum.

This axiom is called the **completeness axiom**. It ensures that the number line has no gaps. Let us think about that a bit.

Fist let us understand that $\mathbb{Q}$ satisfies all axioms discussed so far besides the completeness axiom. Can you find a set in $\mathbb{Q}$ that is bounded but does not have a supremum in $\mathbb{Q}$?

Let us use the completeness axiom to show that square roots of positive real numbers exist. The use of the completeness axiom to prove the existence of certein objects, is typical for analysis.

Theorem 1.1 (Existence of square roots)**.**

Proof. (Sketch)

1.3 Summary of the Axioms for $\mathbb{R}$

As I have stated before, the axioms of the real numbers that we have discussed in Section 1.2, can be clustered in categories. You can find this clustering below.

<table>
<tr><td colspan="2">Field axioms: There are two operations (addition, multiplication) which satisfy:</td><td rowspan="3">Field axioms</td><td rowspan="6">Ordered field axioms</td><td rowspan="7">Complete ordered field axioms</td></tr>
<tr><td>Axioms of addition
Associative law
Commutative law
Existence of the zero
Existence of the negative</td><td>Axioms of multiplication
Associative law
Commutative law
Existence of the one ($\neq 0$)
Existence of the inverse ($\neq 0$)</td></tr>
<tr><td colspan="2">Distributive law</td></tr>
<tr><td colspan="3">Order axioms: there are some elements that are denoted as positive ($x > 0$) such that the following axioms are satisfied</td></tr>
<tr><td colspan="3">Trichotomy: For all elements x and y in $\mathbb{R}$ exactly one of the following is true:
$x < y, \quad x = y, \quad y < x$</td></tr>
<tr><td colspan="3">$x < y$ and $y < z \Rightarrow x < z$
$x < y \quad \Rightarrow \quad x + z < y + z$ for all $z \in \mathbb{R}$
$x < y$ and $z > 0 \Rightarrow xz < yz$.</td></tr>
<tr><td colspan="4">Completeness axiom:
Every bounded set $A \subseteq \mathbb{R}$ has a supremum.</td></tr>
</table>

Table 1.2: Schematic classification of the introduced axioms of the set of the real numbers $\mathbb{R}$.

1.4 Further notes on inequalities

> I love inequalities. So if somebody shows me a new inequality, I say: "Oh, that's beautiful, let me think about it," and I may have some ideas connected to it.
>
> – Louis Nirenberg (Canadian/US Mathematician)

In this section, we extend the *Ordering axioms (II)* by some results that we will need in our subsequent work. In fact, inequalities are the bread and butter of an analyst.

1.4.1 A first example

Consider the following problem:

> Find all possible real values of x such that $\dfrac{1}{x} < x < 1$.

Suppose someone writes this:

How many things can you find that are wrong with this argument?

How would you go about to solve it?

Solving inequalities graphically. It often helps to draw a picture to set yourself in the right set of mind to see how to attack a problem. It falls upon you to use that strategy for problem solving and understanding, not your lecturer.

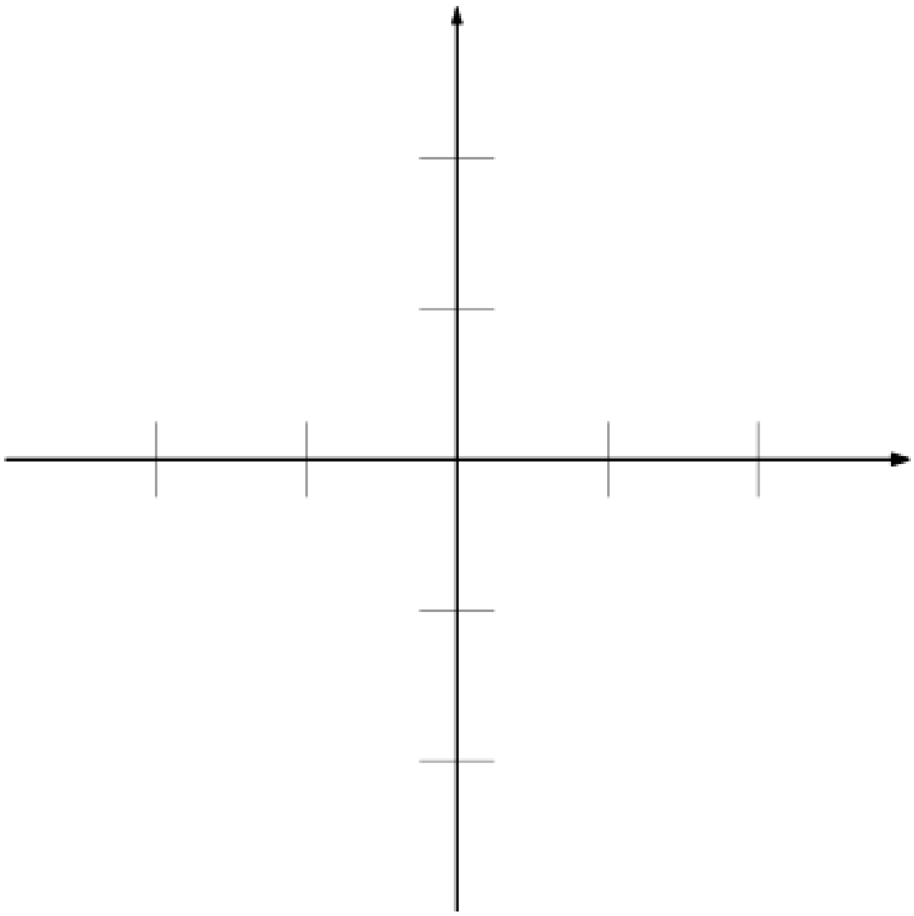


Figure 1.6: Graphical solution of $\frac{1}{x} < x < 1$.

1.4.2 A second example

If $\dfrac{3}{x^2+2} > 1$, what can we say about x?

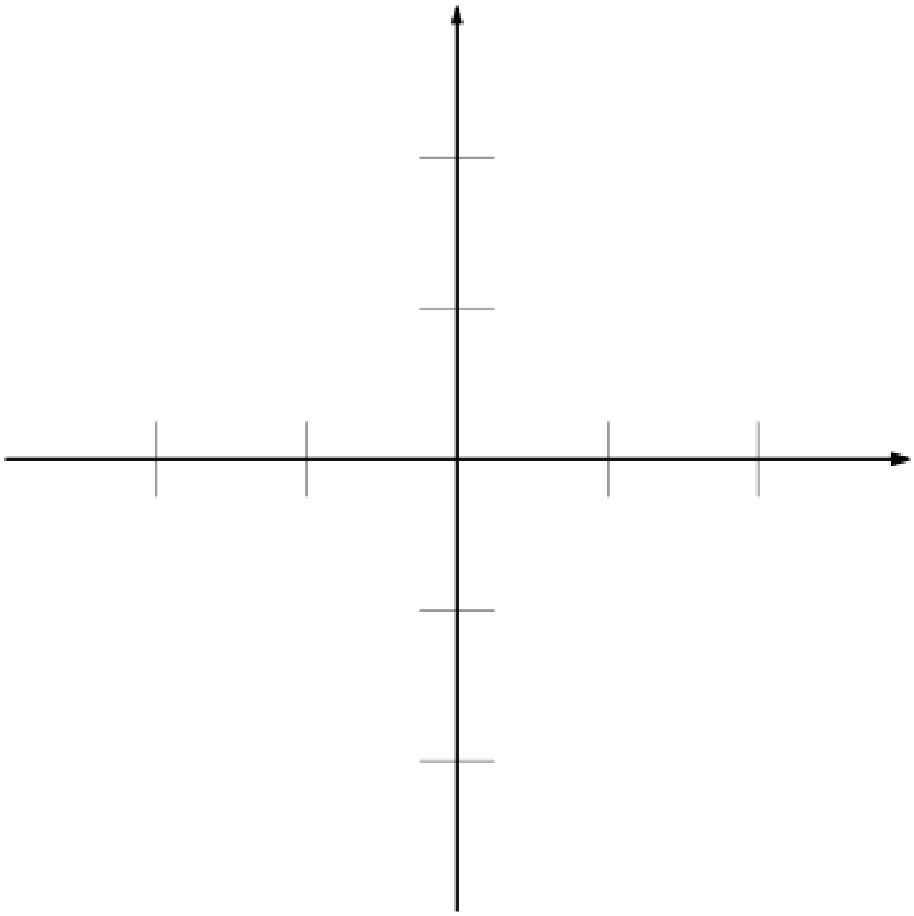


Figure 1.7: A portion of the graph related to the inequality $\frac{3}{x^2+2} > 1$.

Case analysis

Let us consider the following question:

> Which values of x satisfy the inequality $\dfrac{x-1}{x+1} > 0$?

A picture/diagram/graph can be very helpful for constructing proofs and helping readers understand them. However, a picture on its own is not a proof! Figures 1.7 and 1.6, for instance, represents only some parts of the relevant graphs (the bits where it goes off to infinity are not shown). While we feel we know what the graph does in the unshown parts, this picture does not prove anything about them.

To work algebraically, a *proof by cases* can be useful. For instance, it would be good to write something algebraic and unwaffley to answer the question above. However, we cannot just multiply by $x+1$. Why not

We can, however, split an argument into different cases:

With some informal reasoning we can work out what the graph must look like:

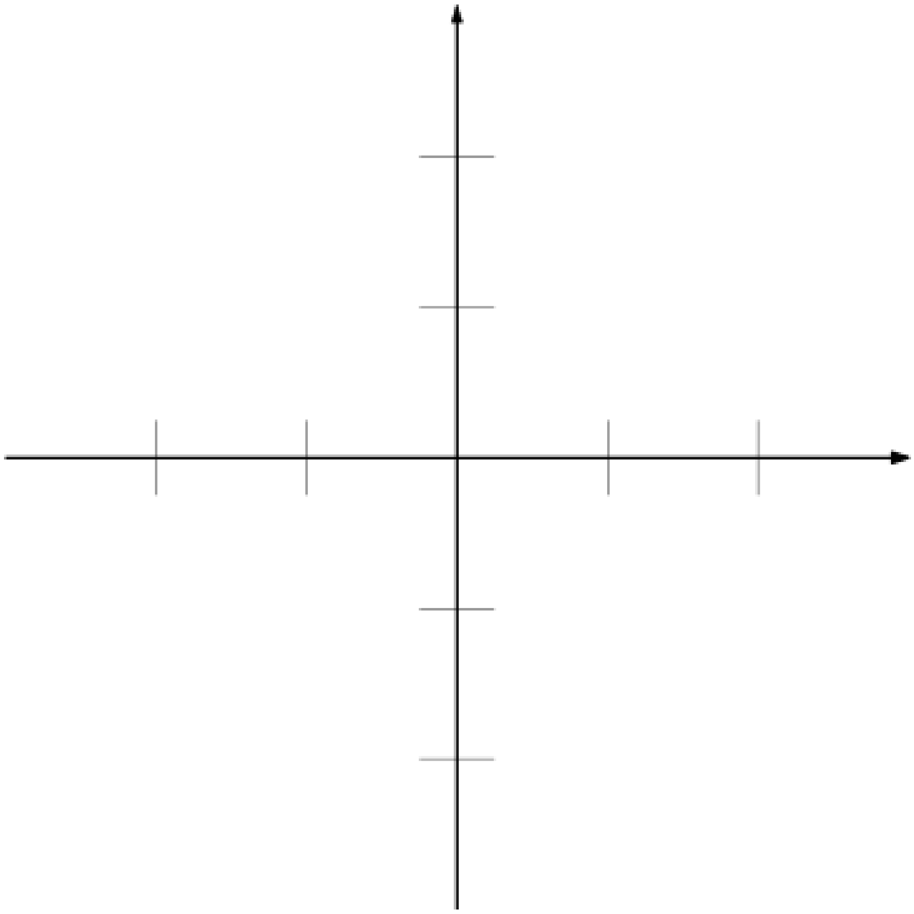


Figure 1.8: A portion of the graph related to the inequality $\dfrac{x-1}{x+1} > 0$.

This is good practice but it involves a bit of hand-waving. You will learn more about graphing in the module *Mathematical Methods*, and you will learn to work more formally with limits and derivatives in later Analysis modules. I suggest that you use Mathematica, Wolfram alpha, or GeoGebra to do graphs related to the problem sheets.

1.5 Absolute values

1.5.1 Defining absolute value

You are probably familiar with the notation $|x|$, as in $|3| = 3$, $|-2| = 2$, $|1+i| = \sqrt{2}$. Formally, we use this definition:

Definition 1.4 (Absolute Value[a]).
Let $x \in \mathbb{R}$. Then the absolute value (or modulus) of x is defined as

$$|x| := \begin{cases} x & x > 0 \\ 0 & : \ x = 0 \\ -x & : \ x < 0 \end{cases} .$$

[a] This notation was introduced by Karl Weierstraß in 1859.

Does this correspond to the way you think about absolute values?

Check that the relationships all work as you would expect.

1.5.2 Properties of absolute values

Theorem 1.2 (Properties of the absolute value).
For any $x, y \in \mathbb{R}$, it holds

(i) $|x| = \max\{x, -x\}$,

(ii) $-|x| \leq x \leq |x|$,

(iii) $|x|^2 = |x^2|$,

(iv) $|xy| = |x||y|$, *and*

(v) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, *if* $y \neq 0$.

Here is another opportunity to practise proving ‘obvious’ things. For example, we can prove (ii). by starting from the definition and using a proof by cases:

Proof.

1.5.3 Interpretation of absolute values

The absolute value $|x|$ of a number $x \in \mathbb{R}$ can be seen as the distance of x to 0. We can generalise this and introduce a notion of distance on $\mathbb{R}$.

How would you define the distance between two real numbers a and b? Let us draw a picture and think.

Let us now give a precise definition:

Definition 1.5 (Distance on $\mathbb{R}$).
We call *the* ***distance*** *between two real numbers* a, $b \in \mathbb{R}$.

Think about the notion of distance. What properties should it have?

Theorem 1.3 (Properties of the distance on $\mathbb{R}$).

The distance of two real numbers a, $b \in \mathbb{R}$ has the following three properties:

1.

2.

3.

1.5.4 Absolute values and inequalities

Inequalities involving absolute values can be difficult to handle. Here are some systematic ways of approaching them:

1. Think of them as representing intervals:

2. Try squaring. For example:

3. Work by cases, as above.

Chapter

2 Sequences

During the semester, you will be expected to spend about an hour between Monday and Wednesday learning new material by reading or working on short problems. The required reading will be indicated in environments as the following.

Reading 1. *This week, you will read from 2.1 to including Section 2.2.1 below, thinking carefully about the text and the questions it contains. At the start of Wednesday's lecture you will test your understanding of this material by working individually and in small groups on associated activities. The end of the required reading is indicated in the text at the end of Section 2.2.1.*

Remark 2.1. *Sections 2.1 and 2.2 are adapted from Sections 5.1 to 5.3 of [1] – they are almost identical so if you have the book you could read that instead.*

2.1 What is a sequence?

2.1.1 Sequences as lists

A sequence is an infinite list, technically an infinite tuple, of numbers, like this:

$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ \ldots$$

or like one of these:

$$1,\ \tfrac{1}{3},\ \tfrac{1}{9},\ \tfrac{1}{27},\ \tfrac{1}{81},\ \tfrac{1}{243},\ \ldots$$
$$1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots$$

An integral part of Analysis I is the study of various properties of real sequences and their relationships. To think flexibly about those relationships, it helps to be aware of some different ways of representing sequences and some advantages and disadvantages of those representations. Even with this simple list representation, there are a few things to notice.:

- First, the list has a comma between each pair of sequence terms and another after the last term that is explicitly listed. This is just notational convention, but it is the kind of thing that looks professional if you get it right.
- Second, the list ends with an ellipsis[1] – a set of three dots. This is a proper punctuation mark, and here it means *and so on forever*. It is important to include the ellipsis – otherwise a mathematically educated reader will assume that the list stops at the last stated term, which is inappropriate because in Analysis the word *sequence* always refers to an infinite sequence. This is not the case in everyday life, where the word 'sequence' might refer to a finite list. As with all definitions in undergraduate mathematics, you are free to think that you prefer the everyday interpretation, but you will have to adhere to the convention in your studies.
- Third, the sequence is infinite *only in one direction*. For instance, this is not a sequence:

[1] ellipsis – The omission from speech or writing of a word or words that are superfluous or able to be understood from contextual clues represented by three dots.

$$\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots$$

Another informal way to say this is that a sequence must have a first term. It might seem odd to remark on this, but some situations tempt students to allow sequences to be infinite *in both directions*.

- Finally, the sequences above follow obvious patterns, but that is not a necessary feature. An infinite list of randomly generated numbers would be a perfectly good sequence. Of course, it would be difficult to work with, so in practice you will mostly see sequences that follow some sort of pattern. But general theorems about sequences apply to all sequences that satisfy their premises, not just to those that are expressible using nice formulas.

Remark 2.2. *Though the list is sometimes used in introductory courses, we will not use it here very often as it is extremely ambiguous. Usually, it is understood that there is some rule that one should find in continuing sequences as*

$$1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0, \ldots$$

and that is what we shall assume for the moment. However, the continuation is by no means unique and therefore the usefulness of this representation is limited. In fact there are infinitely many ways of continuing the sequence in a meaningful way:

$$\begin{aligned} a_n \ &:= \ -\frac{4}{315}n^7 + \frac{2}{5}n^6 - \frac{232}{45}n^5 + 35n^4 - \frac{6028}{45}n^3 \\ &\qquad + \frac{1428}{5}n^2 - \frac{32432}{105}n + 128, \end{aligned}$$

and, alternatively,

$$\begin{aligned} a_n \ &:= \ \frac{43}{13440}n^8 - \frac{1289}{10080}n^7 + \frac{687}{320}n^6 - \frac{14161}{720}n^5 \\ &\qquad + \frac{68367}{640}n^4 - \frac{502883}{1440}n^3 + \frac{2229449}{3360}n^2 - \frac{553963}{840}n + 257 \\ &\ \vdots \end{aligned}$$

2.1.2 Representing sequences

Taking into account Remark 2.2, we introduce the main representation of real sequences that we will use in Analysis I. We will mainly represent sequences by formulas. The first sequence in the list above, assuming that we extend it straightforwardly, might be specified by writing this:

> Let (a_n) be the real sequence defined by $a_n := 2n$ for all $n \in \mathbb{N}$.

Think about the link between such a specification and the fact that a sequence must have a first term. The set $\mathbb{N}$ of natural numbers is the set $\{1, 2, 3, 4, \dots\}$, so this specification yields $a_1 = 2$, $a_2 = 4$, and so on; there is no a_0 or a_{-1}. Note also that a_n denotes the nth element of the sequence and (a_n), $(a_n)_n$, or $(a_n)_{n\in\mathbb{N}}$ denote the whole sequence. Keep in mind that a_n and (a_n) are very different – a_n is a single number and (a_n) is an infinite list[2] of numbers – so make sure you write the one you intend. An alternative notation for the whole sequence is $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n\in\mathbb{N}}$.

Remark 2.3. *The notation $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n\in\mathbb{N}}$, can be confused with sets and therefore will not be used in this notes. However, it is an often used notation in text books. What is the problem with confusion with sets? Remember that for sets*

$$\{1,\ 2,\ 3,\ 1,\ 2,\ 3, \dots\} = \{1,\ 2,\ 3\}$$

and

$$\{1,\ 2,\ 3\} = \{2,\ 3,\ 1\} = \{3, 2, 1, 1\}.$$

[2] A more technical term is tupel. For that see you Linear Algebra module.

To abbreviate further we can write a formula in the brackets, as in sentences like these:

> Consider the sequence $(2n)_{n\in\mathbb{N}}$.
>
> The sequence $\left(\dfrac{1}{3^{n-1}}\right)_{n\in\mathbb{N}}$ tends to zero as n tends to infinity.

The longer formulation is still useful for clarity, however, and we might need it if different terms are specified differently. For instance, the sequence

$$1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots$$

could be specified like this:

> Let (x_n) be the sequence defined by $x_n := \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$.

This is just one sequence, so do not be tempted to think of it as 'two sequences' because of the way it is written. The formula gives a single value for each of x_1, x_2, x_3 and so on as usual.

Here are two more sequences, represented using both formulas and lists. Which formula goes with which list?

$$1, 1, 2, 2, 3, 3, 4, 4, \ldots \qquad\qquad 1, 3, 2, 4, 3, 5, 4, 6, \ldots$$

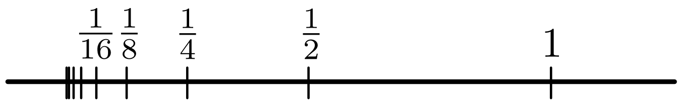
$$b_n := \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} \qquad\qquad c_n := \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n+4}{2} & \text{if } n \text{ is even} \end{cases}$$

Sequences can also be represented graphically.

One standard graphical representation is a number line, and for some sequences, such as

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \dots$$

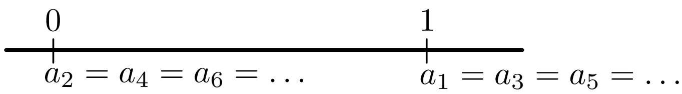
this works quite well:



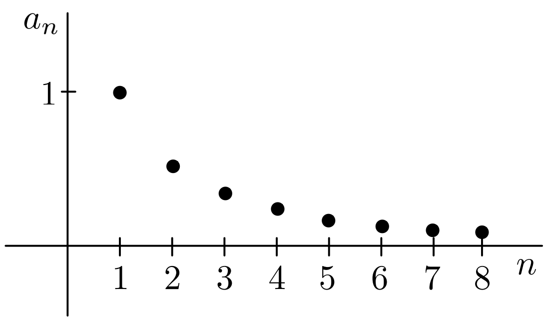
Notice, though, that this diagram does not explicitly represent the order of the terms. Thus, to read it, we have to impose some extra knowledge about which label refers to the first term, which to the second term, and so on. This makes number lines pretty useless for a sequence like

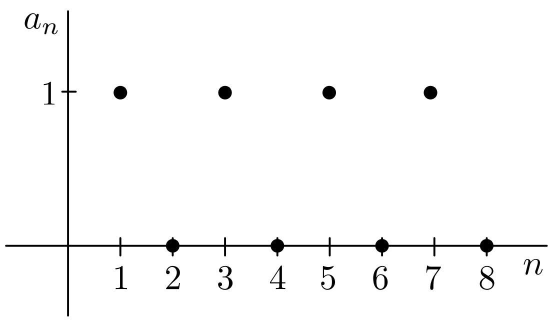
$$1, 0, 1, 0, 1, 0, \dots,$$

although we could add accompanying labels to explain what is going on:



An alternative is to use an extra dimension, graphing a_n against n:





It is appropriate to use dots rather than curves for sequence graphs because each sequence is defined only for natural number values; there is no $a_{\frac{3}{2}}$, for instance. Notice also that this kind of graph uses an axis to explicitly represent the n values as well as the a_n values, so it gives a sense of the long-term behaviour of the sequence. That is handy because the long-term behaviour is often what we're interested in. What would graphs look like for the sequences (b_n) and (c_n) defined as above?

Graphs are also useful for thinking about a mathematical link to the concept of function: a sequence is technically *a function from the natural numbers* $\mathbb{N}$ *(or* $\mathbb{N}_0$*) to the reals* $\mathbb{R}$. Indeed, it might be defined as such at the beginning of an Analysis course. This probably sounds a bit unnatural compared with thinking of a sequence as an infinite list, but you should be able to see why it is reasonable by looking at the graph and by considering that every element $n \in \mathbb{N}$ has a corresponding sequence term a_n.

In school, you probably wrote $f(x)$ for functions f of a variable x. Thus, should not sequences be written $a(n)$ if they are functions? Yes, in some sense. However, as every field, mathematics has history and the terminology bears this history. That is something you have to live with. The subscript notation is *standard* for sequences.

2.2 Sequence properties

2.2.1 Monotonic sequences

The various representations listed above can be useful for thinking about sequence properties. For instance, a sequence might be *increasing*, or *decreasing*, or *bounded*, or *convergent*.

- What do you think these words mean?
- How would you explain their meanings to someone else?
- How would you formulate corresponding mathematical definitions using appropriate notation?

Look away from the Lecture Notes and try this now.

If you gave that a serious go, it should be apparent that, although your intuitive understanding might feel strong, it can be challenging to capture it in a coherent sentence. Awareness of this should put you in the right frame of mind for serious study of the definitions formulated by mathematicians.

Here are the definitions for *increasing* and for *decreasing*.

Definition 2.1 (Increasing sequence)**.**
A sequence (a_n) of real numbers is said to be ***increasing***[a] *iff*

$$\forall n \in \mathbb{N} \ : \ a_{n+1} \geq a_n.$$

[a]Sometimes people say non-decreasing.

Definition 2.2 (Decreasing sequence)**.**

A sequence (a_n) *of real numbers is said to be* ***decreasing***[a] *iff*

$$\forall n \in \mathbb{N} \, : \; a_{n+1} \leq a_n.$$

[a]Sometimes people say non-increasing.

These sound straightforward, but it is surprisingly difficult to think about how they combine. To see what I mean, consider these sequences. Would you say that each one is increasing, decreasing, both, or neither? We will assume that the sequences are continued in the 'obvious' way.

$1, 0, 1, 0, 1, 0, 1, 0, \ldots$

$1, 4, 9, 16, 25, 36, 49, \ldots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots$

$1, -1, 2, -2, 3, -3, \ldots$

$3, 3, 3, 3, 3, 3, 3, 3, \ldots$

$1, 3, 2, 4, 3, 5, 4, 6, \ldots$

$6, 6, 7, 7, 8, 8, 9, 9, \ldots$

$0, 1, 0, 2, 0, 3, 0, 4, \ldots$

$10\frac{1}{2}, 10\frac{3}{4}, 10\frac{7}{8}, 10\frac{15}{16}, \ldots$

$-2, -4, -6, -8, -10, \ldots$

Almost everyone gets some of these wrong. So have another look, checking carefully against the definitions.

Here are the answers.

$1, 0, 1, 0, 1, 0, 1, 0, \ldots$	*neither*
$1, 4, 9, 16, 25, 36, 49, \ldots$	*increasing*
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots$	*decreasing*
$1, -1, 2, -2, 3, -3, \ldots$	*neither*
$3, 3, 3, 3, 3, 3, 3, 3, \ldots$	*both*
$1, 3, 2, 4, 3, 5, 4, 6, \ldots$	*neither*
$6, 6, 7, 7, 8, 8, 9, 9, \ldots$	*increasing*
$0, 1, 0, 2, 0, 3, 0, 4, \ldots$	*neither*
$5, \frac{15}{2}, \frac{35}{4}, \frac{75}{8}, \ldots$	*increasing*
$-2, -4, -6, -8, -10, \ldots$	*decreasing*

Were you right? Even when told to be careful, many Analysis I students get at least one wrong. Most, for instance, want to classify

$$1,\ 0,\ 1, 0,\ 1,\ 0,\ 1,\ 0,\ \ldots$$

as both increasing and decreasing, and almost everyone wants to classify

$$3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ \ldots$$

as neither increasing nor decreasing. This is not surprising because these are perfectly natural interpretations. But they are based on everyday intuition, not on the mathematical definitions.

To understand the first case it helps to think about local versus global properties. When people say that the sequence $1, 0, 1, 0, 1, 0, 1, 0, \ldots$ is both increasing and decreasing, they are usually thinking about local properties. They see the sequence as starting at 1, then decreasing, then increasing, then decreasing, then increasing, and so on. But they should be thinking about a global property, because the definition of *increasing* is a universal statement: it says that for *every* $n \in \mathbb{N}$, the relation $a_{n+1} \geq a_n$ holds true. That certainly is not true for this sequence. Indeed it fails rather badly. There are infinitely many values of n for which a_{n+1} is not greater than or equal to a_n. For instance, $a_2 < a_1$, and $a_4 < a_3$, and so on. So this sequence does not satisfy the definition of increasing. Similarly, it does not satisfy the definition of decreasing. So, mathematically speaking, it is neither increasing nor decreasing.

To understand the second case it is necessary to be careful about the inequality. To satisfy the definition of *increasing*, each term must be *greater than or equal to* its predecessor. If every term is equal to its predecessor, that is enough. This might seem weird, but the definition is reasonable because it is simple and because it means that sequences like

$$6,\ 6,\ 7,\ 7,\ 8,\ 8,\ 9,\ 9,\ \ldots$$

get classified as increasing. It also works well within the theory of Analysis I, because it lends itself to simple theorem statements – many theorems that apply to increasing sequences in general apply to constant sequences in particular.

That said, mathematicians also use these definitions:

Definition 2.3 (Strictly increasing sequence)**.**

*A real sequence (a_n) is said to be **strictly increasing** iff we have*

$$\forall n \in \mathbb{N} \ : \ a_{n+1} > a_n.$$

Definition 2.4 (Strictly decreasing sequence)**.**

*A real sequence (a_n) is said to be **strictly decreasing** iff we have*

$$\forall n \in \mathbb{N} \ : \ a_{n+1} < a_n.$$

Exercise 2.1. *Think about how these apply to the above listed sequences too.*

The final thing to know about the properties *increasing* and *decreasing* is that they are also associated with this definition:

Definition 2.5 (Monotonic sequence)**.**

*A real sequence (a_n) is **monotonic**[a] iff it is increasing or decreasing.*

[a]People sometimes use the word *monotone* instead of *monotonic*.

Students sometimes get confused about this because of the word 'or'. In everyday English, 'or' has two distinct meanings[3] and we are adept at using context and emphasis to work out which is intended. One meaning is *inclusive*, and is used when we mean one thing or the other or both, as in:

[3]This is not the same in all languages—some have different words for inclusive and exclusive or.

> Students wishing to study Applied Statistics in year 3 should ensure that they take Statistical Methods or Introduction to Mathematical Statistics in year 2.

The other meaning is *exclusive*, and is used when we mean one thing or the other but not both, as in:

> Your lunch voucher entitles you to an ice cream or a slice of cake.

To avoid ambiguity in mathematics, we choose one meaning and stick to it, and the interpretation we use is the inclusive one. So this definition means that a sequence is monotonic if it is increasing or decreasing or both and, from the list, these sequences are classified as monotonic:

$$1, 4, 9, 16, 25, 36, 49, \ldots$$

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{7}, \tfrac{1}{8}, \ldots$$

$$3, 3, 3, 3, 3, 3, 3, 3, \ldots$$

$$6, 6, 7, 7, 8, 8, 9, 9, \ldots$$

$$5, \tfrac{15}{2}, \tfrac{35}{4}, \tfrac{75}{8}, \ldots$$

$$-2, -4, -6, -8, -10, \ldots$$

If you need to, please revisit Sections 0.2 and 0.4.

2.2.2 Bounded sequences

When mathematicians give a definition, they really mean it. Specifying properties precisely means that everyone knows what everyone else is talking about, and knows exactly which objects are included when someone states a theorem or gives a proof. Sometimes – as in the case of monotonic sequences – the mathematical use of a word is different from the everyday use of a word. So sometimes your intuition will not correspond with the definitions in the way you expect. Be alert to such cases and make sure that you update your intuitive understanding to reflect the formal knowledge.

Here are some more definitions. Fill in those that are not complete.

Definition 2.6 (Bounded above)**.**
A real sequence (a_n) is bounded above iff there exists a $u \in \mathbb{R}$ such that

$$a_n \leq u$$

for all $n \in \mathbb{N}$.

Definition 2.7 (Upper bound)**.**
*A number $u \in \mathbb{R}$ is an **upper bound** for a real sequence (a_n) iff*

$$a_n \leq u$$

for all $n \in \mathbb{N}$.

Definition 2.8 (Bounded below)**.**
*A real sequence (a_n) is said to be **bounded below** iff*

Definition 2.9 (Lower bound)**.**

A number $l \in \mathbb{R}$ *is a* ***lower bound*** *for a real sequence* (a_n) *iff*

Definition 2.10 (Bounded sequences)**.**

A real sequence (a_n) *is* ***bounded*** *iff it is bounded above and bounded below.*

Proposition 2.1 (Alternative definition of boundedness)**.**

A real sequence (a_n) *is said to be bounded iff there exists a positive constant* M *such that*

$$\forall n \in \mathbb{N} \; : \; |a_n| \leq M.$$

It can be useful to think about definitions in terms of our other representations for sequences:

For each of the following, give an example (written in any representation you wish to use) or state that this is impossible.

1. A sequence that is bounded above but not below.

2. A sequence that has neither an upper bound nor a lower bound.

3. A monotonic sequence that has neither and upper bound nor a lower bound.

4. A decreasing sequence that is bounded below.

5. A strictly decreasing sequence that is bounded below.

6. A bounded, monotonic sequence.

7. A bounded, non-monotonic sequence.

8. A sequence that is not strictly increasing and is not bounded above.

9. A sequence that is neither increasing nor decreasing and is not bounded above.

10. A sequence that is both increasing and decreasing.

2.3 Subsequences

2.3.1 Subsequences definition and practice

What do you think a subsequence is? Try to state it precisely.

Definition 2.11 (Subsequence (Student version))**.**

Now let us give the precise definition.

Definition 2.12 (Subsequence)**.**

Exercise 2.2. *Let (a_n) be the sequence defined by $a_n := \frac{n}{2}$ for all $n \in \mathbb{N}$, so*

$$(a_n) = \left(\frac{1}{2},\, 1,\, \frac{3}{2},\, 2,\, \frac{5}{2},\, \ldots\right).$$

Write down the first five terms of each of these subsequences:

$$(a_{n+4})_{n\in\mathbb{N}} =$$

$$(a_{3n-1})_{n\in\mathbb{N}} =$$

$$(a_{n^2})_{n\in\mathbb{N}} =$$

$$(a_{2^n})_{n\in\mathbb{N}} =$$

2.3.2 Sub-sequences and boundedness

True or false?

Every subsequence of a bounded sequence is bounded.

Theorem 2.1 (Boundedness of sub-sequences of bounded sequences)**.**

Proof.

2.3.3 Sub-sequences and monotonicity

Consider the sequence (a_n) given by $a_n := \cos(n)$, $n \geq 1$.

In fact, this sequence never repeats itself. Can you work out why?

True or false?

> The sequence $\Big(\cos(n)\Big)_{n\in\mathbb{N}}$ has a monotonic subsequence.

True or false?

> Every sequence has a monotonic subsequence.

Clever device:

> Let (a_n) be a sequence. We will call a_f a *floor term* of (a_n) if and only if $\forall n \geq f$, $a_n \geq a_f$.

Figure 2.1: An example of floor terms in a sequence.

Theorem 2.2.

Proof.

2.4 Tending to infinity

You probably have an informal idea of what it should mean for a sequence to tend to infinity. You have also thought about this if you worked through the Problem Sheets.

Can you work out a definition of what it means for a sequence (a_n) of real numbers to tend to $+\infty$?

Definition 2.13 (Sequence tending to infinity (Student's version))**.**

2.4.1 Definition for tending to infinity

Definition 2.14 (Sequence tending to infinity)**.**

Remark 2.4. *In a slight abuse of notation and definition, we say sometimes that a sequence has the limit $+\infty$ or $-\infty$ and use the symbols*

$$(a_n) \to +\infty, \quad \lim_{n\to+\infty} a_n = +\infty, \quad (a_n) \to -\infty, \quad \lim_{n\to+\infty} a_n = -\infty.$$

Fill in this table. Tick all the columns that apply.

sequence	increasing	strictly increasing	$\to +\infty$	bounded below
$1, 4, 9, 16, 25, 36, 49, 64, \ldots$				
$1, -1, 2, -2, 3, -3, 4, -4, \ldots$				
$1, 3, 2, 4, 3, 5, 4, 6, \ldots$				
$6, 6, 7, 7, 8, 8, 9, 9, \ldots$				
$0, 1, 0, 2, 0, 3, 0, 4, \ldots$				

2.4.2 Proving that sequences tend to infinity

Some thoughts about strategy

To prove that a sequence tends to infinity, we need to prove that it satisfies the definition. That is, we need to prove that for any given C there in an index such that all terms of the sequence are greater than C for larger indices.

We will start with a simple example: Consider the sequence given by $a_n := \frac{n}{2}$ for all $n \in \mathbb{N}$. First, we notice that this sequence is strictly increasing. This is helpful because if we can find one term for which $a_n > C$ holds, all the later terms will have that property too. Thus, all we need is to establish when $\frac{n}{2}$ is greater than C. This is not hard: we need $n > 2C$.

Claim

Proof.

Mathematicians like to generalise, and a straightforward generalisation suggests itself here.

Theorem 2.3.

Proof.

2.4.3 A (first) comparison test

Let us investigate what happens if we consider a sequence (a_n) with $(a_n) \to +\infty$ and a sequences (b_n) which is controlled by (a_n) by

$$\forall n \in \mathbb{N} \;:\; a_n \leq b_n.$$

Theorem 2.4 (Comparison theorem for sequences tending to $+\infty$)**.**
Let (a_n) and (b_n) be real sequences. Suppose that $(a_n) \to +\infty$ and that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $b_n \geq a_n$. Then, $(b_n) \to +\infty$.

How would you go about proving that?

1. Write down the definitions involved.

2. Write down what you have to prove.

3. Try to go from one to the other.

Proof of Theorem 2.4.

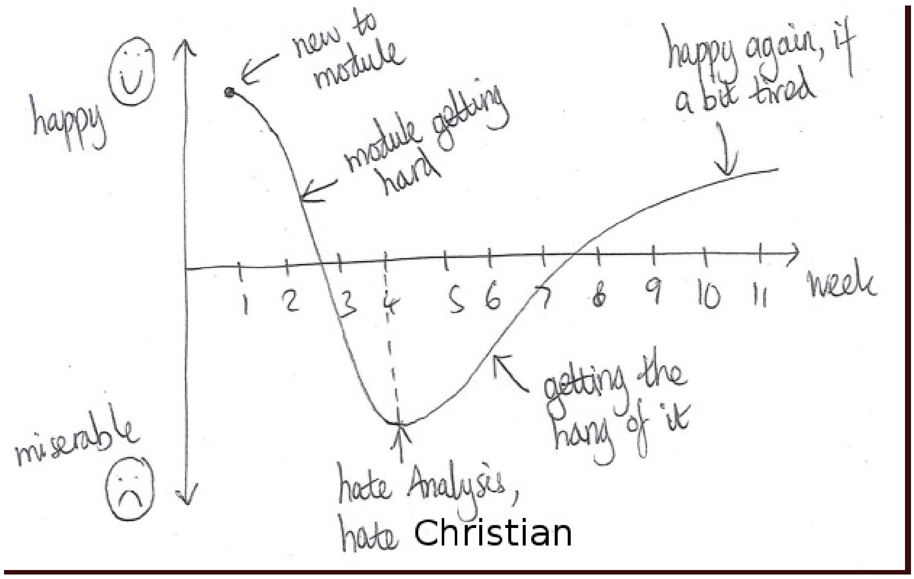
Remark 2.5. *The proof above uses of a good trick. We know that one property holds for all* $n \geq N_1$ *and that another one holds for all* $n \geq N_2$*. We would like both properties to hold at once, but we do not know anything about the relationship between* N_1 *and* N_2*. We can get around this by considering the maximum of* N_1 *and* N_2*. For any number bigger than this maximum, we can be sure that both properties hold. This is a trick that we will use a number of times in Analysis I, and then also in Analysis II; look out for it.*

2.4.4 A comment on difficult mathematics

This is the point at which the module gets difficult.

The definition for tending to infinity (like a couple of others in this module) has three nested quantifiers: it goes '*for all …there exists …for all …*'. In everyday life, few statements are this logically complicated. Hence, no-one has a lot of practice at thinking about them, and few are any good at it at first. This means that if you get confused about it at some point, you are normal. However, you need to constantly work on your understanding of the statements.

The module will stay this difficult. The content will not get *more* difficult, but there will be more of it every week. This means that, during this module, the mood in the class will go like this:



If you know this is going to happen, you should be fine. All it means is that you have to be prepared to keep going, even when you find it challenging. **If you give up, the module will wipe the floor with you.** If you follow the advice in this week's reading, you will end up liking Analysis a lot.

Reading 2. *This weeks reading is the text* ***The analysis experience*** *that you got printed and can also find on the Homework panel on LEARN. It is modified from Chapter 4 of [1].*

2.5 The sequence $(x^n)_{n\in\mathbb{N}}$

2.5.1 Question

For what real values of x does the sequence $(x^n)_{n\in\mathbb{N}}$ tend to infinity?

We will prove this using the comparison test from Section 2.4.3 and Bernoulli's inequality:

> For $x \in \mathbb{R}$, $x \geq -1$ and $n \in \mathbb{N}$, we have $(1+x)^n \geq 1 + nx$.

Exactly how will we use these things, do you think?

2.5.2 Bernoulli's inequality

We will prove Bernoulli's inequality using *proof by induction*. Here is my explanation of how this will apply here.
First, note that for a fixed value of x, the claim holds for all $n \in \mathbb{N}$. So, for a fixed x, we are really looking at a claim about infinitely many propositions:

$P(1)$: $(1+x)^1 \geq 1 + 1x$

$P(2)$: $(1+x)^2 \geq 1 + 2x$

$P(3)$: $(1+x)^3 \geq 1 + 3x$

$P(4)$: $(1+x)^4 \geq 1 + 4x$, and so on forever.

When we construct a proof by induction, we first prove that $P(1)$ is true. This is sometimes referred to as establishing the *base case*.

Then we do something clever. We do not prove any of the remaining propositions directly. Instead we prove that *if* $P(k)$ is true, *then* $P(k+1)$ is true as well. This is sometimes referred to as proving the *induction step*.

This works because when we have done both parts we have proved $P(1)$, and $P(1)$ implies $P(2)$, and $P(2)$ implies $P(3)$, and so on. So we get an infinite chain of true propositions:

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \ldots$$

Remark 2.6. *If you need to brush up on the Principle of Induction, you should revisit the notes of the Module* Mathematical Thinking.

Exercise 2.3. *Prove by induction that*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Exercise 2.4. *Can you work out a formula for*

$$\sum_{k=1}^{n} k^2$$

or

$$\sum_{k=1}^{n} k^3$$

and prove them by induction?

Theorem 2.5 (Bernoulli's Inequality[a]).

Let $x \in \mathbb{R}$, $x \geq -1$ *and* $n \in \mathbb{N}$. *Then,*

$$(1+x)^n \geq 1 + nx.$$

[a] Jacob Bernoulli first published the inequality in his treatise *Positiones Arithmeticae de Seriebus Infinitis* (Basel, 1689), where he used the inequality often.

How would you go about proving that theorem?

1. A theorem involving an $\forall$-statement over the natural numbers lends itself to a proof by induction.

2. Remind yourself how induction works and attempt a proof.

Proof of Theorem 2.5.

Corollary 2.1.

Let $x \in \mathbb{R}$, $x > 1$. *Then*

$$\lim_{n \to +\infty} x^n = +\infty.^{a}$$

[a] In other notation, we might write $(x^n) \to +\infty$, $n \to +\infty$ if $x \in \mathbb{R}_{>1}$.

Proof.

2.6 Convergent sequences

2.6.1 Convergence definition and meaning

Informally, we say that a sequence converges if its terms approach a finite limit. These are all ways of saying and writing the same thing:

- (a_n) converges to a.
- (a_n) tends to a (as n tends to infinity).
- $(a_n) \to a$.
- $a_n \to a$ as $n \to \infty$.
- $\lim\limits_{n \to +\infty} a_n = a$.

Before we proceed, draw a picture of $(a_n) \to a$. Can you formalise a definition?

Definition 2.15 (Convergence of sequences (Student's version))**.**

Definition 2.16 (Convergence of sequences)**.**

Let us write the definition in plain English:

Notes:

1. This definition is similar in structure to that for tending to infinity. Look back and make sure you can see this.
2. Any n_0 that works will do – the definition does not say anything about finding the 'first' one.
3. We do not talk about what happens 'at infinity' at all.
4. The terms do **not** have to get closer to a each time, e.g.

Definition 2.17 (Divergent sequence)**.**

Notice that a sequence can diverge in all sorts of ways:

2.6.2 Proving that sequences tend to zero (or not)

Definition 2.18 (Nullsequence).

Example: a sequence that tends to zero

As you would expect, the sequence $\left(\frac{1}{\sqrt{n}}\right)$ converges to zero.

To prove this it helps to do a bit of thinking first:

Claim:

Proof.

Example: a sequence that does not tend to zero

Also as you would expect, the sequence $0, 1, 0, 1, 0, 1, 0, 1, \ldots$ does not converge to zero.

Again, some first thinking:

Claim

The sequence (a_n) given by $a_n := \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

does not converge to zero.

Proof.

2.6.3 Reciprocals of sequences that tend to infinity

One fairly obvious result is that if (a_n) tends to $+\infty$, then $\left(\dfrac{1}{a_n}\right)$ tends to zero. How will we prove this?

Theorem 2.6.

Let (a_n) be a real sequence and suppose that $(a_n) \to +\infty$. Then,

$$\lim_{n\to+\infty} \frac{1}{a_n} = 0.$$

Proof

Question:

1. Is the converse of Theorem 2.6 true?

2. Is there a true partial converse?

2.6.4 Convergent sequences and absolute values

Question:

What word goes in the next Theorem? Iff or then?

Theorem 2.7.
Let (a_n) *be a null-sequence* $(|a_n|)$ *is a null-sequence.*

Proof

Question:

What word goes in the next Theorem? $\Leftrightarrow$, $\Rightarrow$, or $\Leftarrow$?

Theorem 2.8.

Let $a \in \mathbb{R}$, $a \neq 0$. *Then,* $(a_n) \to a$ $\qquad\qquad$ $(|a_n|) \to |a|$.[a]

[a] Note that the only reason for excluding $a = 0$ is that in that case an even stronger statement is true.

Proof.

Before Wednesday you should read the self-explanation training provided in booklet form and apply the training to the theorem and proof below, making notes on anything you do not understand. You should also apply it to theorems and proofs from earlier in the module, particularly some of the harder ones from today.

Note 2.1. *If you are struggling to understand the definition of convergence for sequences, download the sequences chapter from How to Think about Analysis, which can be found on the Lecture Notes panel on Learn, and study Sections 5.5 and 5.6 (in either order).*

2.6.5 Arithmetic for convergent sequences

Theorem 2.9 (Arithmetic properties of limits).
Let (a_n) and (b_n) two convergent real sequences with $a_n \to a$ and $b_n \to b$. Then, we have

1. *The sequence $(a_n + b_n)$ converges with*

$$\lim_{n\to+\infty} \left[a_n + b_n\right] = a + b. \quad \textit{(sum rule)}$$

2. *The sequence $(a_n b_n)$ converges with*

$$\lim_{n\to+\infty} a_n b_n = ab \quad \textit{(product rule)}$$

3. *For $c \in \mathbb{R}$, the sequence (ca_n) converges with*

$$\lim_{n\to+\infty} ca_n = ca. \quad \textit{(constant multiple rule)}$$

4. *If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then the sequence $\left(\frac{a_n}{b_n}\right)$ converges with*

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{a}{b}. \quad \textit{(quotient rule)}$$

Remark 2.7. *The above theorem ensures that the set of convergent real sequences* (a_n) *are a vector space with the component-wise addition* $(a_n) + (b_n) = (a_n + b_n)$ *and* $\lambda(a_n) = (\lambda a_n)$. *See also your Lecture Notes from Linear Algebra.*

Proof of the sum rule for convergent sequences

Let $\varepsilon > 0$ be arbitrary.
Then $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2}$
and $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2}$.
Now, let $n_0 = \max\{N_1, N_2\}$.
Then $\forall n \geq n_0$,

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \text{ by the triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

So $\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N}$

$$|(a_n + b_n) - (a + b)| < \varepsilon \quad \forall n \geq n_0.$$

So $(a_n + b_n) \to a + b$ as required. $\square$

Notice that in the proof of the sum rule, we split up $|a_n - a + b_n - b|$ into two pieces, and set up conditions under which each would be less than $\frac{\varepsilon}{2}$. For the product rule, we would like to do something similar with $|a_n b_n - ab|$.

- Why do we want to do that?
- What is the problem?
- To get around this problem, we can use a clever trick:

- How is this used in the proof below?

Let us now put things together.

Proof of the product rule for convergent sequences

2.6.6 Examples using convergence theorems

We can combine the rules we have proved so far with our existing knowledge to derive lots of new results. For example:

Important note: For many people, the experience of school mathematics has led to a ***belief*** *that mathematics is basically a bunch of procedures to learn. Hence, when they see things like this, they think 'oh, right, now that we are doing calculations I can ignore the theorems'.* ***This is NOT true****. In Analysis (and in much other undergraduate mathematics) it is the definitions, theorems, and proofs that we care about. You will be tested on your knowledge of these and on your ability to work with them.*

2.6.7 Sandwich theorem

Let (a_n), (b_n), and (c_n) be real sequences. Suppose that $(a_n) \to a$ and $(b_n) \to a$ and that

$$a_n \leq c_n \leq b_n, \quad \forall n \in \mathbb{N}.$$

- What can we conclude?
- How do you think a proof might go? What will we assume and how will we work from that to what we want to prove? Might it help to draw a diagram?
- Do we actually need the condition to hold $\forall n \in \mathbb{N}$?
- Did you use this idea in your own thinking about the problems for this Week?

Theorem 2.10 (Sandwich Theorem)**.**

Proof.

2.6.8 Shift theorem

Theorem 2.11 (Shift Theorem)**.**
Let (a_n) be a real sequence and suppose $N \in \mathbb{N}_0$. Then $(a_n) \to a$ iff $(a_{N+n}) \to a$.

Remark 2.8. *This theorem yields that a finite number of elements of a sequence do not influence the convergence at all and can be dropped.*

Proof

The Proof is for you to figure out on the Problem Sheet. You might want to add it to the notes here.

2.6.9 Limits are unique

The theorem below is the kind of thing that makes students think mathematicians are a bit bonkers. Why bother saying such an obvious thing? But this theorem can be proved using the definition of convergence, so we do that in order to see how it fits into the theory. Also, the theorem is not at all obvious. If you learn a little more you will see that this theorem is due to $\mathbb{R}$ and its nice properties. There are other spaces, that you will see in the module Topology, in which limits are not necessarily unique. In fact, this proof will use a common tactic for proving uniqueness: assume that there are two of the object in question, and prove that they must be the same.

Theorem 2.12 (Uniqueness of limits)**.**
A real sequence can not converge to more than one limit, we say the limit is unique.

Proof.

2.6.10 Convergence and boundedness

What symbol goes in the gap in this theorem? $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$?

Theorem 2.13.

Let (a_n) be a real sequence. Then,

$$(a_n) \text{ is convergent} \qquad\qquad (a_n) \text{ is bounded.}$$

Some thoughts about the strategy of the proof:

Proof.

2.6.11 Boundedness and limits

Suppose that (a_n) is a real sequence with $(a_n) \to a$ and $A \leq a_n \leq B$ for all $n \in \mathbb{N}$. What can we say about a?

Suppose that (a_n) is a real sequence with $(a_n) \to a$ and $A < a_n < B$ for all $n \in \mathbb{N}$. What can we now say about a?

How do you think we might prove a result about this?

Theorem 2.14.

Proof.

2.6.12 Recursively-defined sequences

Theorem 2.12 is useful because it can help us to decide between two possible limits.

Consider the recursively-defined sequence given by $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 2}$.

Write out a few terms to get a feel for this.

Now (a_n) converges (this is *not* trivial – we will come back to it later).

With this assumption, we can work out what it converges to:

2.7 Standard limits

In this section, we will look at some sequences for which our results about shifts, products and so on do not immediately apply.

2.7.1 Sequences of the form (x^n)

Theorem 2.15.

Let $x \in \mathbb{R}$ be arbitrary.

$$\lim_{n\to\infty} x^n = \begin{cases} +\infty & : \\ 1 & : \\ 0 & : \end{cases}$$

Otherwise, the sequence $(x^n)_{n\in\mathbb{N}}$ has no limit.

Reading 3. *Sections 2.7.1 to 2.7.3 form this week's reading. You should come to the next lecture prepared to answer questions and engage in activities on all the recent material. As you do this reading, remember your self-explanation training, and remember to keep your* Questions about Analysis *list.*

Proof (by cases)

If $x = 1$ then $x^n = 1$ $\forall n \in \mathbb{N}$ so $(x^n) \to 1$.

If $x > 1$ then we can use Bernoulli's inequality:

Bernoulli's inequality states that if $y > -1$ then $(1 + y)^n \geq 1 + ny$ $\forall n \in \mathbb{N}$.

Now $x > 1 \Rightarrow x - 1 > 0 > -1$.

So rewriting $x = 1 + (x - 1)$ and using Bernoulli's inequality gives

$$x^n = (1 + (x - 1))^n \geq 1 + n(x - 1) \quad \forall n \in \mathbb{N}.$$

Now $(n(x - 1)) \to +\infty$ by an earlier result because $x - 1 > 0$.

So $(1 + n(x - 1)) \to +\infty$ by the comparison test.

So $(x^n) \to +\infty$, also by the comparison test.

If $|x| < 1$ and $x \neq 0$ then $\dfrac{1}{|x|} > 1$ so $\left(\left(\dfrac{1}{|x|}\right)^n\right) \to +\infty$ by the previous case.

So $\left(\dfrac{1}{\left(\frac{1}{|x|}\right)^n}\right) \to 0$ by the result about reciprocals, i.e. $(|x|^n) \to 0$.

So $(x^n) \to 0$ by the result about absolute values of sequences tending to zero.

If $x = 0$ then $x^n = 0 \ \forall n \in \mathbb{N}$ so $(x^n) \to 0$.

Notice that we are not done yet!

If $x < -1$ then $x^n < -1$ when n is odd and $x^n > 1$ when n is even.

So (x^n) cannot tend to $+\infty$ or $-\infty$,

and if $(x^n) \to l \in \mathbb{R}$ then we would have $l \leq -1$ and $l > 1$, which is impossible.

So (x^n) has no limit in this case.

2.7.2 The sequence (n^α)

We have

Theorem 2.16.

Let $\alpha \in \mathbb{R}$ be arbitrary. Then,

$$\lim_{n \to +\infty} n^\alpha = \begin{cases} +\infty & : \\ 1 & : \\ 0 & : \end{cases}$$

Proof.

You will do the proof on the Problem Sheet.

You might then want to write it here.

2.7.3 Sequences of the form $\left(\dfrac{n^\alpha}{x^n}\right)$

We will start by showing that the particular sequence $\left(\frac{n^2}{2^n}\right)$ tends to zero. We will compare this sequence with $\left(\left(\frac{3}{4}\right)^n\right)$, the sequence with *common ratio* $\frac{a_{n+1}}{a_n}$ equal to $\frac{3}{4}$.
We know $\left(\left(\frac{3}{4}\right)^n\right) \to 0$ because it is a sequence of the form (x^n) with $|x| < 1$.
Our sequence does not have a common ratio, but it does *eventually* have ratio $\leq \frac{3}{4}$, so we will be able to use the sandwich theorem to prove that it tends to zero too.
We will tackle this by examining the ratio $\frac{a_{n+1}}{a_n}$:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}\left(1+\frac{1}{n}\right)^2.$$

What happens as n increases?
$\dfrac{a_{n+1}}{a_n}$ gets close to $\frac{1}{2}$, and certainly below $\frac{3}{4}$. When?

$$\begin{aligned}
\frac{1}{2}\left(1+\frac{1}{n}\right)^2 < \frac{3}{4} \quad &\Leftrightarrow \quad \left(1+\frac{1}{n}\right)^2 < \frac{3}{2} \\
&\Leftrightarrow \quad 1+\frac{1}{n} < \sqrt{\frac{3}{2}} \\
&\Leftrightarrow \quad \frac{1}{n} < \sqrt{\frac{3}{2}} - 1 \\
&\Leftrightarrow \quad n > \frac{1}{\sqrt{\frac{3}{2}} - 1}.
\end{aligned}$$

So $n > 5$ certainly gives $\dfrac{a_{n+1}}{a_n} < \frac{3}{4}$, which we can rewrite as $a_{n+1} < \frac{3}{4}a_n$.
So $a_7 < \frac{3}{4}a_6$, and $a_8 < \frac{3}{4}a_7 < \left(\frac{3}{4}\right)^2 a_6$, and $a_9 < \frac{3}{4}a_8 < \left(\frac{3}{4}\right)^2 a_7 < \left(\frac{3}{4}\right)^3 a_6$, and so on.
In general, $a_{6+n} < \left(\frac{3}{4}\right)^n a_6$.
Now $\left(\left(\frac{3}{4}\right)^n a_6\right) \to 0$.
So $(a_{6+n}) \to 0$. *Why?*
So $(a_n) \to 0$. *Why?*

That is the end of this week's reading. Have you thought about everything properly? Remember that these are your notes so you can write all over them if you like. And did you recognise the reasoning on this page from the Problem Sheet problems?

Notice that in the reasoning on the previous page we proved something about the original sequence (a_n) *by looking not at the individual terms but at the ratio* $\frac{a_{n+1}}{a_n}$*. It is important to keep this straight if you are to understand how ratio tests work. We will come across several in this module.*

Notice also that this is brilliant. If you are thinking about it properly, you should enjoy seeing such a clever argument that makes neat use of many of our previous results.

2.7.4 d'Alembert's ratio test

Theorem 2.17 (Ratio test for sequences)**.**

Suppose that (a_n) *is a real sequence and there is an* $l \in \mathbb{R}$ *such that*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l.$$

. Then:

(i) If $-1 < l < 1$ *then* $(a_n) \to 0$*.*

(ii) If $l > 1$ *and* $a_n > 0 \; \forall n \in \mathbb{N}$ *then* $(a_n) \to \infty$*.*

(iii) If $l > 1$ *and* $a_n < 0 \; \forall n \in \mathbb{N}$ *then* $(a_n) \to -\infty$*.*

(iv) If $l < -1$ *then the sequence neither converges nor tends to* $\pm\infty$*.*

(v) If $l = 1$ *we get no information.*

Proof of (i).

2.7.5 Sequences of the form $\left(x^{\frac{1}{n}}\right)$

What does $\left(3^{\frac{1}{n}}\right)$ converge to?
What about $\left(100^{\frac{1}{n}}\right)$?

Claim

Proof

Claim

Proof

2.7.6 The sequence $\left(n^{\frac{1}{n}}\right)$

Notice that it is less clear what will happen for this sequence.

Perhaps the same type of argument will work again to show that it converges to 1?

We will use a trick:

Theorem 2.18.

Proof.

2.7.7 Sequences of the form $\left((x^n + y^n)^{\frac{1}{n}}\right)$

This seems like such a good idea that we will use it to cover other cases too.
Consider $\left(\left(4^{10} + 2^n\right)^{\frac{1}{n}}\right)$.

2.7.8 Making summaries

Can you fit the contents of the chapter on Sequences on one page, in the form of a list or a concept map?

Chapter

3 Sequences II

In this section, we go away from studying concrete sequences and prove more general results that will prove to be valuable in the future.

3.0.1 Monotonicity, boundedness, and convergence

Does the theorem below seem obvious to you? If it does, you are implicitly assuming completeness – the theorem would not hold if we were working only with rational numbers.

Theorem 3.1.

Every increasing sequence (a_n) that is bounded above is convergent.

Proof.

The proof is for you on the problem sheet to figure out. You might want to add it here. Think about the above comment where the completeness axiom comes in.

From that follows in a straight forward way

Corollary 3.1.

Every decreasing sequence that is bounded below is convergent.

From Theorem 3.1 and Corollary 3.1 follows immediately

Corollary 3.2.

Every monotonic bounded sequence is convergent.

With that, we can state and prove the immensely important

Theorem 3.2 (Bolzano–Weierstrass)**.**

Every bounded sequence has a convergent subsequence.

Proof.

The proof is an exercise. You might want to put it here.

3.1 Cauchy sequences

We now have a test that allows us to establish that a monotonic sequence converges without knowing its limit.

It would be nice to have a similar test for non-monotonic sequences. Would this work?

(a_n) is convergent if $(|a_{n+1} - a_n|) \to 0$.

3.1.1 Cauchy sequences

Definition 3.1 (Cauchy sequence)**.**

Think hard about what this means.

In the definition of convergence, we specify that beyond a certain point in the sequence, all the terms are within distance ε of the limit.

In the definition of a Cauchy sequence, we specify that beyond a certain point in the sequence, all the terms are within distance ε of each other.
Make sure you compare the forms of the two definitions.

Exercise 3.1. *The sequence* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ *is a Cauchy sequence.*

Proof.

3.1.2 Cauchy sequences and convergence

It probably will not surprise you to learn that every convergent sequence is Cauchy, and vice versa. Here are some diagrams to help us think about how to prove this.

Theorem 3.3.

Let (a_n) be a real sequence. Then, (a_n) is convergent iff (a_n) is a Cauchy sequence.

To prove Theorem 3.3, we first prove a lemma:

Lemma 3.1 (Cauchy sequences are bounded)**.**

Let (a_n) be a Cauchy sequence. Then, there exists a non-negative constant C such that

$$|a_n| \leq C \quad \forall n \in \mathbb{N}.$$

Why? What can we do with that?

Proof of Lemma 3.1.

Proof of Theorem 3.3.

[$\Rightarrow$] Let $\varepsilon > 0$ be arbitrary.

Since $(a_n) \to a$, there exists $n_0 \in \mathbb{N}$ s.t. for all $n, m \geq n_0$, $|a_n - a| < \frac{\varepsilon}{2}$ and $|a_m - a| < \frac{\varepsilon}{2}$.

So for all $n, m \geq n_0$ we have

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So (a_n) is a Cauchy sequence.

[$\Leftarrow$] Since (a_n) is a Cauchy sequence, it is bounded by Lemma 3.1. Hence, by the Bolzano-Weierstrass theorem, (a_n) has a convergent subsequence, say (a_{n_i}). Suppose that $(a_{n_i}) \to a$. We will show that $(a_n) \to a$ also.

To this end, let $\varepsilon > 0$ be arbitrary.

Then, since (a_n) is Cauchy, there exists $n_1 \in \mathbb{N}$ such that

$$\forall n, n_i \geq n_1, \ |a_n - a_{n_i}| < \frac{\varepsilon}{2}.$$

Also, since $(a_{n_i}) \to a$, there exists $n_1 \in \mathbb{N}$ such that

$$\forall i \geq n_2, \ |a_{n_i} - a| < \frac{\varepsilon}{2}.$$

Let $n_0 = \max\{n_1, n_2\}$. Then, since $n_i \geq i$, we have

$$\forall n \geq n_0, |a_n - a| = |a_n - a_{n_i} + a_{n_i} - a| \leq |a_n - a_{n_i}| + |a_{n_i} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $(a_n) \to a$, i.e. (a_n) is convergent.

CHAPTER

4 Series

This is my favourite section of the course, because it has some properly weird stuff in it – stuff that will really make you think. As ever, though, we will start with basic notions and work up to more advanced material.

4.1 Introduction to series

4.1.1 Sequences and series are different

A *sequence* is an infinite list of terms.

A *series* is an infinite sum of terms.

Series and sequences are different objects and you should make sure to use the correct terminology for what you want to say.

4.1.2 Representing series

To represent series, we often use 'Σ notation':

We can use different letters, e.g.

Because we will want to study infinite sums via finite sums, I will try to stick to using i and n in this way. It is easy to get mixed up, though, so when you write an expression like this, check to make sure that you have used the letters in the way you intended.

4.1.3 Question for discussion

Consider this argument about an infinite sum:

Let $S = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots.$

Then $xS = \quad x + x^2 + x^3 + x^4 + x^5 + \ldots.$

So $S - xS = 1$

i.e. $(1 - x)S = 1.$

So $S = \dfrac{1}{1-x}.$

Does this work for every value of x?

What exactly goes wrong in some cases?

Remark 4.1.

To make the above precise, we should work with the finite sum $\sum\limits_{k=0}^{n} x^k = S_n$*, work out a formula for the sum* S_n *and then consider the resulting sequences* S_n *for* $n \to +\infty$*.*

4.2 Series convergence

4.2.1 Partial sums

Definition 4.1 (Partial sum)**.**

Notice that (s_n) is a sequence, where:

4.2.2 Convergence and divergence

Definition 4.2 (Convergence/divergence of series)**.**
Let $\sum_{i=1}^{\infty} a_i$ *be a series and* (s_n) *be the associated sequence of partial sums. Then:*

Definition 4.3 (Sum of a series)**.**
If a series $\sum_{i=1}^{+\infty} a_i$ *converges, then we call the limit of the partial sums the sum of the series.*

Remark 4.2.
The language might be a bit confusing here. We are really interested in whether or not the whole series adds up to a single finite number. But, because the series is infinite, we approach the question via the sequence of partial sums. This leads us to describe the behaviour of series in terms of convergence. You need to know the formal definition in terms of partial sums, obviously, but here is a short summary of the conceptual information:

- *We say that a series converges if it adds up to a finite number;*
- *We say that it diverges if it does not.*

Remark 4.3.
Because of this, Remark 4.2, when we work with series, we often have two different sequences floating around:

- *an original sequence* (a_i)*;*
- *a sequence* (s_n) *made up of sums of these terms.*

Before you start on anything, make sure you are clear about the relationship between these things, and that you know which one(s) you should be thinking about.

4.2.3 A condensed definition

We can reformulate our definition of convergence without explicitly naming the partial sums:

Definition 4.4 (Equivalent to Def. 4.2)**.**

As ever, we will use whichever formulation makes our arguments clearer and/or more straightforward.

4.2.4 A Cauchy criterion

Definition 4.5 (Cauchy Criterion of Convergence)**.**

4.2.5 An example of convergence

An auxiliary result:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = 1 - \frac{1}{n+1}.$$

Tp prove this by induction is left to you as an exercise. (Can you generalise the above result by telescoping an arbitrary null-sequence?)

4.2.6 An example for divergence

4.3 Simple tests for convergence

4.3.1 Shift rule for series

We would like to prove the theorem below. To do so, we will need to sort out the relationship between the partial sums for the two series

$$\sum_{i=1}^{+\infty} a_i \quad \text{and} \quad \sum_{i=1}^{+\infty} a_{N+i}.$$

If (s_n) is the sequence of partial sums for the first series, is it true that (s_{N+n}) is the sequence of partial sums for the second?

Theorem 4.1 (Shift rule for series)**.**
Let $N \in \mathbb{N}$. Then $\sum_{i=1}^{+\infty} a_i$ converges if and only if $\sum_{i=1}^{\infty} a_{N+i}$ converges.

Proof.

4.3.2 Comparison test for series

Theorem 4.2 (Comparison test for series).

Suppose that $\forall n \in \mathbb{N}$, $0 \leq a_n \leq b_n$. *Then*

Proof.

A check on understanding: Let (a_n) and (b_n) be sequences and suppose that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$.

- Suppose that $\sum a_n$ converges. What can we say about $\sum b_n$?
- Suppose that $\sum b_n$ diverges. What can we say about $\sum a_n$?

4.4 Geometric series

Sections 4.4.1 and 4.4.2 form this week's reading. They are adapted from Sections 6.4 and 6.6 of [1].

4.4.1 Convergence of geometric series

Working with partial sums converts a question about series into a question about sequences. This allows us to give a precise answer to the earlier question about geometric series, while also generalizing a bit to answer this question:

> For what values of a and x is it true that $a + ax + ax^2 + ax^3 + \ldots = \dfrac{a}{1-x}$?

Using partial sums means that we can apply the familiar argument to the partial sum s_n, which is finite so that we do not run into problems with infinite or undefined sums. Then we can ask what happens to s_n as n tends to infinity, in effect turning the question about an infinite sum into a question about finite sums and a limit. Here is a whole argument, presented as a theorem and proof.

Theorem 4.3.

Let $a \in \mathbb{R}$. Then, we have that

$$a + ax + ax^2 + ax^3 + ax^4 + \ldots = \frac{a}{1-x}$$

if and only if $|x| < 1$.

Proof:

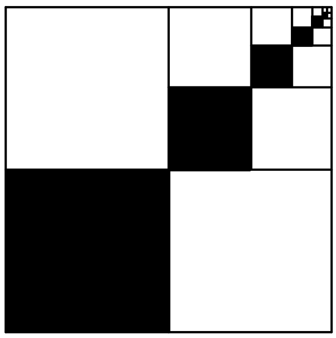
$$\begin{array}{llcl}
\text{Let} & s_n & = & a + ax + ax^2 + \ldots + ax^{n-1}. \\
\text{Then} & xs_n & = & \quad ax + ax^2 + \ldots + ax^{n-1} + ax^n. \\
\text{So} & s_n - xs_n & = & a - ax^n, \\
\text{i.e.} & (1-x)s_n & = & a - ax^n. \\
\text{So} & s_n & = & \dfrac{a(1-x^n)}{1-x}.
\end{array}$$

Now (x^n) converges if and only if $|x| < 1$.

So (s_n) converges if and only if $|x| < 1$.

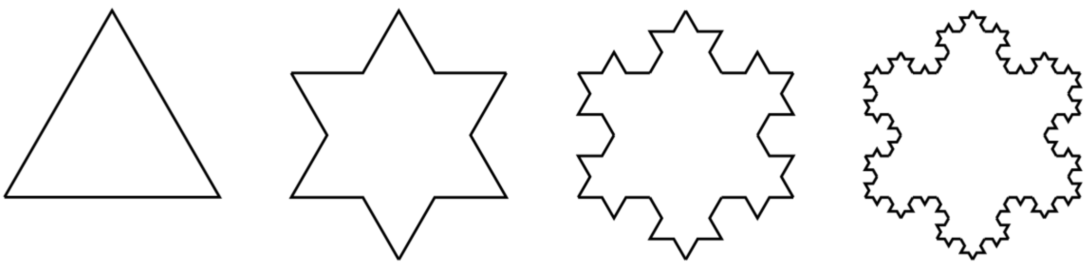
In such cases, $(x^n) \to 0$ so $(s_n) \to \dfrac{a}{1-x}$.

With this established, I think it is fun to look at visual representations for certain sums. For instance, imagine that the area of the whole square below is 1. What is the area of the biggest black square? And the next biggest one? How does the picture illustrate the sum?



$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \ldots = \frac{1}{3}$

Another image that you might have seen is the *Koch snowflake*, which is constructed iteratively by taking an equilateral triangle and adding on three triangles to construct a six-pointed star, then smaller triangles and so on. If the area of the original triangle is 1, what is the area of the star? And what is the area at the next iteration? What geometric series does this construction correspond to, and what is its sum (the area of the limiting shape)? And, for something a bit weirder, what is the perimeter of that shape?



Keep reading – Section 4.4.2 is included.

4.4.2 Ratio test for series

Remember how the ratio test for sequences worked, and how the proof involved comparing with a geometric sequence (see Section 2.7.4)? Now that we have cleared up the question about convergence for geometric *series*, we can prove a very similar ratio test for convergence of other series.

Note that in the work below we are working with series with positive terms only. Series with some negative terms can have peculiar properties so we will come back to those later.

Theorem 4.4 (Ratio test for series)**.**
Let (a_n) be a sequence of positive numbers and assume that

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l.$$

Then:

(i) If $l < 1$, then the series $\sum a_n$ converges.

(ii) If $l > 1$ (including $l = \infty$), then the series $\sum a_n$ diverges.

Remark 4.4. *The ratio test says noting at all if $l = 1$. The series $\sum_{n=1}^{+\infty} a_n$ may converge or diverge.*

We will do two things here: apply this test to a specific series and examine a proof. We will apply it to the series below. What do you think? Does this series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \frac{5^2}{2^5} + \dots$$

To find out using the ratio test, we need to consider $\frac{a_{n+1}}{a_n}$. The form of the series terms means that things cancel when we do this:

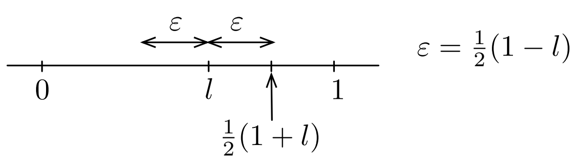
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2.$$

Now as $n \to +\infty$, we get that $\left(1 + \frac{1}{n}\right)^2 \to 1$ and so $\frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2}$. This is a limit $l < 1$ so the ratio test tells us that the series converges. That is all there is to applying the ratio test. But people sometimes find it confusing, I think because it gives information about the series via the limit of the sequence of ratios of its terms, which is obviously a complicated chain of reasoning. Check that you can see what I mean, then try applying the ratio test to find out about the convergence or otherwise of the series

$$\sum_{n=1}^{+\infty} \frac{2^n}{n!} = \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \dots .$$

Why does the cancelling work even better in this case? And why is it important to remember that we use $\frac{a_{n+1}}{a_n}$, not $\frac{a_n}{a_{n+1}}$?

To understand why the ratio test works, we need a proof. Below I state the test again, together with a proof for part 1. I like this proof because it is a nice example of theory building: it uses the definition of convergence for the sequence $\frac{a_{n+1}}{a_n}$ (see Section 2.6.1), the result about convergence for geometric series (Section 4.4.1) and the comparison test and the shift rule (Section 4.3.1). It also cleverly constructs a number less than 1 by using the fact that $l < 1$. This diagram will help you to see how:



With that in mind, try reading the proof (do not forget the self-explanation training).

Theorem 4.5 (Ratio test for series)**.**
Suppose that $a_n > 0\ \forall n \in \mathbb{N}$ and that $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$. Then:

1. *If $l < 1$ then $\sum a_n$ converges.*
2. *If $l > 1$ (including $l = \infty$) then $\sum a_n$ diverges.*
3. *If $l = 1$, the test is inconclusive.*

Proof of part 1:

Suppose $a_n > 0\ \forall n \in \mathbb{N}$ and $\left(\frac{a_{n+1}}{a_n}\right) \to l$, $l < 1$.

Then, using $\varepsilon = \frac{1}{2}(1-l)$ in the definition of $\left(\frac{a_{n+1}}{a_n}\right) \to l$,

$\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$,

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \tfrac{1}{2}(1-l) \quad \Rightarrow \quad \frac{a_{n+1}}{a_n} < l + \tfrac{1}{2}(1-l) = \tfrac{1}{2}(1+l) < 1.$$

This means that $\forall n \geq n_0$, $a_{n+1} < \frac{1}{2}(1+l)a_n$.

In particular,

$$a_{n_0+2} < \left(\tfrac{1}{2}(1+l)\right)^2 a_{n_0}$$

and

$$a_{n_0+3} < \tfrac{1}{2}(1+l)a_{n_0+2} < \left(\tfrac{1}{2}(1+l)\right)^3 a_{n_0}$$

and, by induction,

$$a_{n_0+n} \leq \left(\tfrac{1}{2}(1+l)\right)^n a_{n_0} \quad \forall n \in \mathbb{N}.$$

Now $\sum \left(\frac{1}{2}(1+l)\right)^n a_{n_0}$ converges because it is a geometric series with common ratio less than 1.

So $\sum a_{n_0+n}$ converges by the comparison test for series.

So $\sum a_n$ converges by the shift rule for series.

As usual, I would advise imagining that you are explaining the proof to someone else. Where, if anywhere, do you get stuck? Make a couple of notes on that and you will be ready to listen for my explanation. If you did not get stuck, can you adapt the argument to prove part 2?

That's the end of this week's reading.

4.5 Combining series

In undergraduate mathematics you will often find that when a new type of mathematical object is introduced, we spend some time thinking about how we might combine objects of this type. Which of these would be meaningful, and under what conditions?

- adding two series together;
- multiplying a series by a constant;
- multiplying two series together.

Theorem 4.6.

Proof.

4.6 Series of the form $\sum \frac{1}{n^\alpha}$

4.6.1 A short recapitulation of what we know

4.6.2 The series $\sum \frac{1}{n^\alpha}$

For which values of α do we now know about the behaviour of the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$?

You will finish our work on series of this form by clearing up the cases for which $1 < \alpha < 2$. You might want to answer the following questions on blank paper so that on the next page you can write a full argument using the key points.

1. Write out the first fifteen terms of the series $\sum \dfrac{1}{n^\alpha}$.

2. Verify that $s_7 - s_3 < \dfrac{4}{4^\alpha}$ and $s_9 - s_4 < \dfrac{5}{5^\alpha}$ and, in general, $s_{2n-1} - s_{n-1} < \dfrac{n}{n^\alpha}$.

3. Write down these inequalities for $n = 2, 4, 8, \ldots, 2^k$ and add them to show that
$$\sum_{i=1}^{2^{k+1}-1} \frac{1}{i^\alpha} < 1 + \frac{2}{2^\alpha} + \frac{4}{4^\alpha} + \frac{8}{8^\alpha} + \ldots + \frac{2^k}{2^{k\alpha}}.$$

4. What is the common ratio in the series $1 + \dfrac{2}{2^\alpha} + \dfrac{4}{4^\alpha} + \dfrac{8}{8^\alpha} + \ldots$?

5. Deduce that
$$\sum_{i=1}^{2^{k+1}-1} \frac{1}{i^\alpha} < \frac{1 - \left(\frac{1}{2^{\alpha-1}}\right)^{k+1}}{1 - \left(\frac{1}{2^{\alpha-1}}\right)}.$$

6. Are the terms of the series $\sum \dfrac{1}{n^\alpha}$ all positive?

7. Is the sequence of partial sums for the series $\sum \dfrac{1}{n^\alpha}$ bounded?

8. What can we conclude about the behaviour of $\sum \dfrac{1}{n^\alpha}$?

9. Summarise the behaviour of $\sum \dfrac{1}{n^\alpha}$ for all possible values of α.

Write up a proof on the next page, thinking carefully about your justifications.

Also, the argument you have constructed uses various ideas that have cropped up in earlier work. How would you describe these ideas and where do they appear?

4.7 The null-sequence test

What properties does (a_n) necessarily have if the series $\sum_{i=1}^{+\infty} a_i$ converges?

Theorem 4.7 (Null-sequence test)**.**
Let the series $\sum_{i=1}^{+\infty} a_i$ *be convergent. Then,* (a_n) *is a null-sequence.*

Proof.

This theorem is called the null sequence test because its contrapositive acts as a test to establish that a series is *not* convergent.
How, exactly?

Note that the converse of this theorem is *not* true. What is everyone's favourite counterexample to show this?

4.8 More sophisticated comparisons

4.8.1 Intuition priming

Do you think these series converge or diverge? What makes you think that?

$$\sum_{n=1}^{\infty} \frac{n^2+5n}{n^3+7}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n+2}$$

We will develop a *limit comparison test* that allows us to formally compare series that are 'like' each other in this way. We will get there in two stages.

4.8.2 Second comparison test for series

Suppose $\forall n \in \mathbb{N}$, $a_n, b_n \geq 0$ and $\exists\, m, M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, and $0 < m < \dfrac{a_n}{b_n} < M$.
What can we say about the two series $\sum a_n$ and $\sum b_n$?

Theorem 4.8 (Second comparison test for series).

A note on logic

Proof.

No reading for this week. Instead, do this:

- *Complete the questions on page 148 and write up a nice proof on the next page;*

- *Complete the proof on this page (using the note about logic on page 152).*

4.8.3 Limit comparison test for series

Theorem 4.9 (Limit comparison test)**.**

Proof.

Example 4.1. *We can apply this test directly to the series* $\sum_{n=1}^{\infty} \frac{n^2+5n}{n^3+7}$. *How?*

4.8.4 A note on tests involving ratios

The limit comparison test makes use of the ratio a_n/b_n, which is a ratio of the corresponding terms of two *different* series.

The ratio test for series makes use of the ratio $\frac{a_{n+1}}{a_n}$, which is a ratio of adjacent terms of the *same* series.

4.9 Series with some negative terms

4.9.1 The series $\sum \frac{(-1)^{n+1}}{n}$

We can sketch a graph showing the sequence of partial sums for the series $\sum \frac{(-1)^{n+1}}{n}$:

It turns out that this series converges to $\ln(2)$ but here we will only prove that it converges.

We will establish that the series converges, which will give us ideas that can be generalised when we work with other *alternating series*.

Claim: The series $\sum \frac{(-1)^{n+1}}{n}$ converges.

Proof.

4.9.2 The weirdest thing in Analysis 1

Consider the series $\sum a_n = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} \dots$

Now consider the series $\sum b_n = 1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \dots$

$\sum b_n$ has all the same terms as $\sum a_n$, just in a different order. Make sure you believe this.

Grouping in threes gives $\sum b_n = 1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \dots$

When confronted with counterintuitive things like the result above, mathematicians do two things:

1. Generalise:

2. Ask why:

We will do both this week.

When does a rearrangement make no difference?

Theorem 4.10.
Let $a_1, \ldots a_n$ be a finite number of real numbers. Then, the sum

$$a_1 + a_2 + \ldots + a_n$$

does not depend on the order of the addition.

Remark 4.5. *Pause here and think for a moment. If someone had told you at the beginning of this module that this would be worth writing down as a theorem, you probably would have thought them crazy. The fact that you can see that this needs to be said is a measure of how much you have learned.*

Remark 4.6. *Note that this theorem could be proved using commutativity of addition and mathematical induction from the axioms on the real numbers. Do it!*

We have already done a lot of work on series with positive terms, so we hope that nothing too weird happens there, either.

Theorem 4.11.

Proof.

Question: Where exactly does this proof use the fact that all the terms are positive?

4.9.3 Alternating series

To understand which other series have the same peculiar behaviour as the series $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$, we will first work with some general properties of *alternating series*.

Theorem 4.12 (Leibniz Criterion/Alternating Series Test)**.**
Suppose that (a_n) *is decreasing and converges to zero. Then* $\sum_{n=1}^{+\infty} (-1)^{n+1} a_n$ *is convergent.*

Remark 4.7. *Notice that the premises of the above theorem mean that* $a_n \geq 0$ $\forall n \in \mathbb{N}$*.*

We can prove this in the same way we proved that $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$ *is convergent on page* *156**. Work out how to generalise that proof and fill this in.* It might help to note that

$$s_{2n+2} = a_1 - a_2 + \ldots + a_{2n-1} - a_{2n} + a_{2n+1} - a_{2n+2},$$

so for example $s_{2n+1} - s_{2n-1} = -a_{2n} + a_{2n+1}$.

Proof.

Applications

For which of these series could we apply the Alternating Series Test to prove convergence? What other results would we need to use in each case?

- $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2}$

- $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\left(\tfrac{3}{7}\right)^n$

- $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^n}{\sqrt{n}}$

- $\displaystyle\sum_{n=1}^{+\infty}\frac{n}{(-2)^n}$

Now think about the premises of the alternating series test. We need both, but why? What might go wrong if one or the other does not hold? Where would the proof of the alternating series test break down?

4.9.4 The series $\sum \dfrac{(-1)^{n+1}}{n^2}$

To understand why some series exhibit such weird behaviour, it will help to understand why this one does not. We could prove that this converges using the alternating series test, but it will help if we also look at an alternative proof. This alternative proof involves defining some new sequences:

Reading 4. *This week's reading starts here and includes this section and Section 4.9.5. Make sure you read in a thoughtful and engaged way – do not forget the self-explanation training.*

Exercise 4.1. *Justify every step and be sure that you understand why every line follows from what has been said before. You might want to look back to Theorem 1.2 for some properties of $|\cdot|$.*

Claim: The series $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2}$ converges.

Proof

Let $a_n = \dfrac{(-1)^{n+1}}{n^2}$.

Note that $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges, i.e. $\displaystyle\sum_{n=1}^{+\infty} |a_n|$ converges.

Define u_n and v_n by $u_n = \frac{1}{2}(|a_n| + a_n)$ and $v_n = \frac{1}{2}(|a_n| - a_n)$.

Then $\forall n \in \mathbb{N}$, $0 \leq u_n \leq |a_n|$ and $0 \leq v_n \leq |a_n|$.

So $\displaystyle\sum_{n=1}^{+\infty} u_n$ and $\displaystyle\sum_{n=1}^{+\infty} v_n$ both converge by the (first) comparison test.

So $\sum_{n=1}^{+\infty} (u_n - v_n)$ converges by the sum rule for convergent series.

But $u_n - v_n = a_n$.

So $\sum_{n=1}^{+\infty} a_n$ converges. □

Here is the thing to notice in order to understand the weird behaviour of some series:

The above proof relies on the fact that $\sum_{n=1}^{+\infty} |a_n|$ converges. Is that the case for $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$?
This distinction motivates the idea of *absolute convergence*.

4.9.5 Absolute convergence

Definition 4.6 (Absolute convergence).
The series $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent if and only if* $\sum_{n=1}^{+\infty} |a_n|$ *is convergent.*

Remark 4.8. *Note that with this definition, a series that is convergent but not absolutely convergent is conditionally convergent. See Definition 4.7. Make a sheet where you write down all the convergence notions for series and examples and non-examples for all of them.*

This definition means that:

- The series $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2}$ is convergent and absolutely convergent;
- The series $\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}$ is convergent but not absolutely convergent.

If you have been reading thoughtfully, it will not surprise you to learn that every absolutely convergent series is convergent, which is stated and proved below. Notice that this proof is a straightforward generalisation of the one you just read. Look back to check.

Theorem 4.13 (Absolute convergent $\Rightarrow$ convergent).
Every absolutely convergent series is convergent.

Proof

Suppose that $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, i.e. $\sum_{n=1}^{+\infty} |a_n|$ converges.

Define u_n and v_n by $u_n = \frac{1}{2}(|a_n| + a_n)$ and $v_n = \frac{1}{2}(|a_n| - a_n)$.

Then $\forall n \in \mathbb{N}$, $0 \leq u_n \leq |a_n|$ and $0 \leq v_n \leq |a_n|$.

So $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ both converge by the (first) comparison test.

So $\sum_{n=1}^{+\infty} (u_n - v_n)$ converges by the sum rule for convergent series.

But $u_n - v_n = a_n$.

So $\sum_{n=1}^{+\infty} a_n$ converges. □

Notice that this result gives us another chance to think about converses:

$$\begin{aligned} \text{absolute convergence} &\Rightarrow \text{convergence} \\ \text{convergence} &\not\Rightarrow \text{absolute convergence} \end{aligned}$$

It is also important in the context of our efforts to understand series because it turns out that if a series is absolutely convergent, rearrangement cannot change its sum.

Theorem 4.14 (Rearrangement of absolutely convergent series)**.**
Suppose that $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent and* $\sum_{n=1}^{+\infty} a_n = A$*. Then, for any sequence* (n_k) *of natural numbers, we have*

$$\sum_{k=1}^{+\infty} a_{n_k} = A.^{a}$$

[a] In other words: any rearrangement of $\sum_{n=1}^{+\infty} a_n$ converges and also adds up to A.

Proof.

Suppose that $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent and $\sum_{n=1}^{+\infty} a_n = A$.

Define u_n and v_n by

$$u_n = \frac{1}{2}(|a_n| + a_n) \quad \text{and} \quad v_n = \frac{1}{2}(|a_n| - a_n).$$

Then $0 \leq u_n \leq |a_n|$, $0 \leq v_n \leq |a_n|$ and, thus $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ are absolutely convergent.

Suppose that $\sum_{n=1}^{+\infty} b_n$ is a rearrangement of $\sum_{n=1}^{+\infty} a_n$.

Define x_n and y_n by

$$x_n := \frac{1}{2}(|b_n| + b_n) \quad \text{and} \quad y_n := \frac{1}{2}(|b_n| - b_n).$$

Then $0 \leq x_n \leq |b_n|$, $0 \leq y_n \leq |b_n|$ and, thus, $\sum\limits_{n=1}^{+\infty} x_n$ and $\sum\limits_{n=1}^{+\infty} y_n$ are absolutely convergent.

We have that $\sum\limits_{n=1}^{+\infty} x_n$ is a rearrangement of $\sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=1}^{+\infty} y_n$ is a rearrangement of $\sum\limits_{n=1}^{+\infty} v_n$. (Is that clear?)

Also $u_n, v_n, x_n, y_n \geq 0$ for all $n \in \mathbb{N}$.

So, by the theorem about rearranging convergent series with positive terms,
$\sum\limits_{n=1}^{+\infty} x_n = \sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=1}^{+\infty} y_n = \sum\limits_{n=1}^{+\infty} v_n$.

Also $\forall n \in \mathbb{N}$, we have

$$a_n = u_n - v_n \quad \text{and} \quad b_n = x_n - y_n.$$

Hence

$$\begin{aligned}
\sum_{n=1}^{+\infty} b_n &= \sum_{n=1}^{+\infty} (x_n - y_n) = \underbrace{\sum_{n=1}^{+\infty} x_n - \sum_{n=1}^{+\infty} y_n}_{\text{Why are we allowed to do that?}} \\
&= \sum_{n=1}^{+\infty} u_n - \sum_{n=1}^{+\infty} v_n \\
&= \sum_{n=1}^{+\infty} (u_n - v_n) = \sum_{n=1}^{+\infty} a_n = A
\end{aligned}$$

$\square$

Notice we have now proved that rearrangements of convergent sequences make no difference to its sum when:

- there are only finitely many terms;
- the terms of a series are all positive;
- a series is absolutely convergent.

4.9.6 Riemann's rearrangement theorem

Now we state (but do not prove) the wonderfully weird Riemann rearrangement theorem.

It makes us forcefully aware that infinite sums a very different from finite sums.

To state the theorem, we introduce one more

Definition 4.7 (Conditional convergence)**.**
A series $\sum_{k=1}^{+\infty} a_k$ *is said to be conditionally convergent if* $\sum_{k=1}^{+\infty} a_k$ *converges but* $\sum_{k=1}^{+\infty} |a_k|$ *diverges.*

Example 4.2. *An example of a conditionally convergent series is the alternating series* $\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}$ *since the harmonic series* $\sum_{k=1}^{+\infty} \frac{1}{k}$ *diverges.*

Remark 4.9. *The sequence* (a_n) *of summands of a conditionally convergent series* $\sum_{k=1}^{+\infty} a_k$ *must have a sub-sequence of just negative elements. Can you make that clear?*

Theorem 4.15 (Riemann's Rearrangement Theorem)**.**
Let (a_n) *be a real sequence and let the series* $\sum\limits_{n=1}^{+\infty} a_n$ *be conditionally convergent. Further, let* $R \in \mathbb{R}$. *Then, there exists a sequence* (n_k) *of natural numbers such that*

$$\sum_{k=1}^{+\infty} a_{n_k} = R.$$

There exists also a sequence (n_k) *of natural numbers such that*

$$\sum_{k=1}^{+\infty} a_{n_k} = +\infty$$

as well as a sequence (n_k) *such that*

$$\sum_{k=1}^{+\infty} a_{n_k} = -\infty.$$

Finally, there exists sequences (n_k) *of natural numbers such that* $\sum\limits_{k=1}^{+\infty} a_{n_k}$ *does not attain any limit.*

4.10 Power series

This final section provides an introduction to power series, which pop up all over mathematics and its applications as physics and engineering. You will study them in more detail in other modules (you have already begun to do so, actually). Here, I want to make sure that you understand what power series are, how they can be studied using techniques we have already seen, and how they relate to functions.

4.10.1 Introduction to power series

Definition 4.8 (Power series)**.**

Notice:

- Each term is of the form $c_n(x-a)^n$, where c_n is a coefficient.
- We usually start at $n = 0$ because this allows us to have a constant term.
- Allowing a constant term is good because a power series is like an infinite polynomial.
- The powers of x or $(x-a)$ give power series their name.

Introductory examples

Here are some familiar power series:

- $\displaystyle\sum_{n=0}^{\infty} x^n$

- $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

- $\displaystyle 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Questions:

1. What are the coefficients c_n in each case?

2. For which values of x does each of the above series converge?

You probably know that

- $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ is the MacLaurin Series of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$;
- $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$ is the MacLaurin Series of $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \cos x$.

Do you really know what that means?

4.10.2 Power series as functions of x

Consider again the series $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$, which converges $\forall x \in \mathbb{R}$.
What does that mean?

Now consider again the series $\sum_{n=0}^{\infty} x^n$.

This too can be thought of as an infinite polynomial in x.

However, it converges only for some values of x, not for all of them.

That should clarify what it means to treat a power series as a function.

But it does not explain how such a function might relate to a familiar one like $g(x) = \cos x$.

We can get to an explanation for that by thinking about partial sums.

For the series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$, the first few partial sums are:

Notice that each of these is a function of x, too. So we can plot a few on the same graph as g. In the diagram on the next page, which graph is which, and what do you notice about improving approximations?

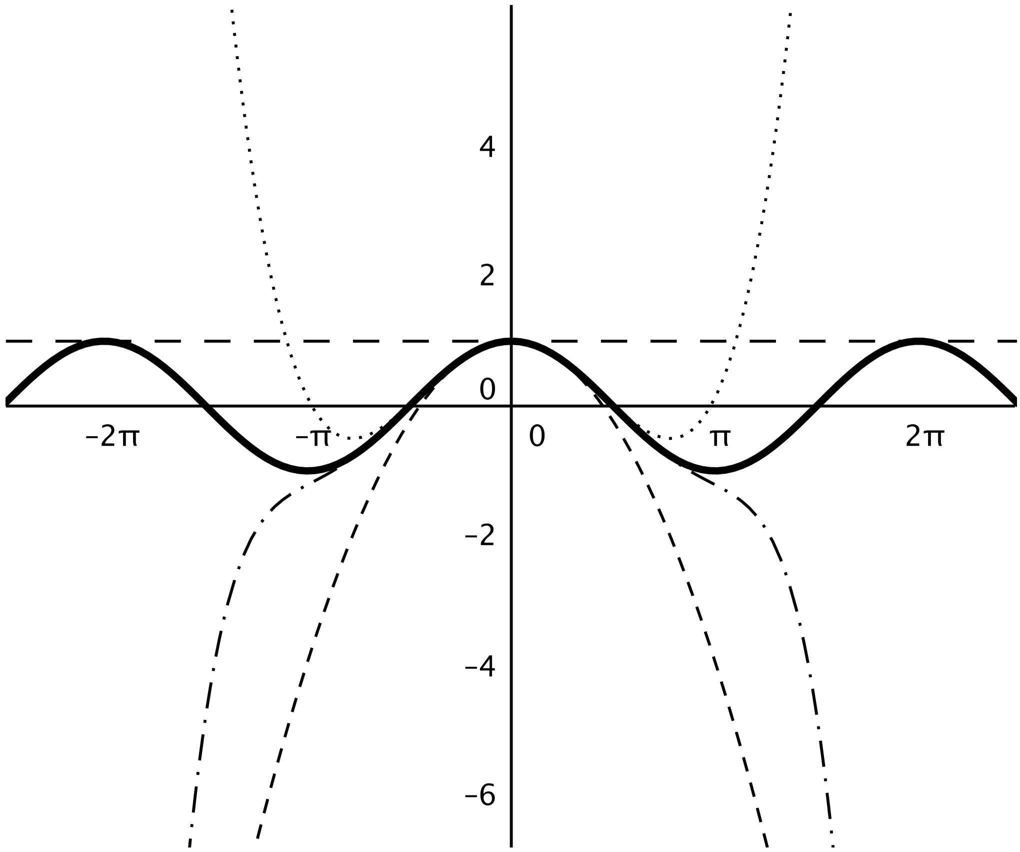


Figure 4.1: Taylor approximations of different degrees of the function $f(x) = \cos(x)$.

In fact, it is more exciting to look at these approximations for higher powers of n. I like to do this using GeoGebra or Wolfram alpha. Here are some instructions so you can have a go at this yourself once you've watched me.

1. In the bottom input line, type $f(x) = \cos(x)$ and hit return.

2. Click the second-from-the-right top button to add a slider. Click on the screen where you want to put it. Call the number n and make it run from 0 to 30 in increments of 1, then click 'apply'.

3. In the bottom line, type Taylorpolynomial[f,0,n] and hit return. This produces the graph of the nth partial sum of the power series approximation to $f(x) = \cos(x)$ about the point 0.

4. Click the left-hand top button to get the pointer, and use it to change n with the slider. This is fun, so don't forget to think about what you're looking at.

5. If you want to zoom out, you can click the right-hand top button to find that option. Just click the magnifying class cursor on the drawing pad to use it.

6. Of course, you can mess around with all the inputs to explore other functions, points and partial sums.

4.10.3 Extending the ratio test

We have established that some power series converge for every $x \in \mathbb{R}$, and some don't. In this section we will explore this idea further. How can we tell for which x a power series converges? Again, it will help to start with some illustrative examples.

Exercise 4.2 (Power series centred at $a = 0.$).
Consider the power series $\sum_{n=0}^{\infty} \frac{5^n x^n}{n+1}$. *Assuming* $x > 0$ *and applying the ratio test:*

What if we allow $x < 0$ so that some of the terms are negative?

Theorem 4.16 (Ratio test).
Let $\sum_{n=1}^{+\infty} a_n$ *be a series with* $a_n \neq 0$. *Suppose that there exists* $l \in \mathbb{R}$ *such that*

$$\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = l$$

Then:

1. *If* $l < 1$ *then the series converges absolutely.*
2. *If* $l > 1$ *(including* $l = \infty$*) then the series diverges.*

Proof. The proof is on the problem sheet for you to figure out. You might want to put it here.

Example: series with some negative terms, centred at 0

Consider the power series $\sum_{n=0}^{\infty} \frac{(-5)^n x^n}{n+1}$. Applying the ratio test:

Does this give you some insight into why we do not have an $l = 1$ case in the ratio test?

Exercise 4.3 (Power series centred somewhere else)**.**

Consider the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2n}$

4.10.4 Radius of convergence

We will not prove the theorem below, but thinking properly about how it relates to the examples in the previous section should convince you that it is true.

Theorem 4.17.

Let $a \in \mathbb{R}$ and (c_n) be a real sequence. Then, for a power series of the form

$$\sum_{n=0}^{+\infty} c_n(x-a)^n,$$

exactly one of the following is true.

1. *The power series converges (absolutely) for all $x \in \mathbb{R}$.*
2. *The power series converges only for $x = a$.*
3. *There exists $R > 0$ s.t. the power series converges (absolutely) for $x \in \mathbb{R}$ with $|x - a| < R$[a] and diverges for $x \in \mathbb{R}$ with $|x - a| > R$[b].*

[a] i.e. $x \in (a - R, a + R)$.
[b] i.e. $x \in (-\infty, a - R) \cup (a + R, +\infty)$.

Definition 4.9 (Radius of convergence)**.**

The number R in Theorem 4.17 is called the radius of convergence *of the power series. In the first case, we set $R = +\infty$ and in the second case $R = 0$.*

Why is it called the radius of convergence? There are no circles here...

If asked to find the radius of convergence, you want the number R.

If asked to find all the values of x for which a power series converges, what else should you do?

4.10.5 Taylor series

Some of you will be familiar with the MacLaurin series from Section 4.10.1, and some might know that we can find the *Taylor series* for a general function f about a point a using this formula:

This formula looks complicated and people often find it intimidating. But it really is not that bad because each term has exactly the same form. That is because the notation $f^{(n)}(a)$ means the nth derivative of f at a. Look carefully to check that you can see that the form is indeed the same each time. (Also, note that $f^{(n)}(a)$ is to be distinguished from $f^n(a)$, which means $f(a)$ raised to the power n. Be careful to write the one you intend.)

To derive the formula, suppose that we can express f as a power series, i.e. by writing

We need to find the coefficients, and one can be identified immediately: setting $x = a$ gives

We can find the other coefficients with some judicious differentiation and substitution. Differentiating both sides gives

and setting $x = a$ gives

Differentiating again gives

and setting $x = a$ again gives

Get the idea? It is worth doing one more:

and setting $x = a$ gives

I did not multiply out the numbers because the structure is easier to see this way. Try a couple more steps and you will see that this leads to

$$c_n = \frac{f^{(n)}(a)}{n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2} = \frac{f^{(n)}(a)}{n!}.$$

This means that the whole series must be the Taylor series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

Now, that is a nice derivation, and people who have done a lot of calculus might have seen it before. But in Analysis we do more than just differentiation and algebra – we think about the conditions under which an argument is valid. This derivation shows that *if* a function is equal to a power series about the point x_0, then that power series must be the Taylor series. But this does not tell us under what conditions the 'if' applies.

We have looked at a couple of cases (you might know some more) in which the full Taylor series is exactly equal to the function for all values of x. But we have also looked at a function for which that is not the case.

If you go through the above process of differentiation and substitution for the function given by $f(x) = \frac{1}{1-x}$ about the point $a = 0$, you will end up with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.$$

But we know that this equality holds only for $x \in (-1, 1)$. The function $f(x) = \frac{1}{1-x}$ is defined unproblematically for lots of other x-values too, but it is not equal to this power series for those values. There also exist functions are not equal to their Taylor series anywhere except at $x = a$. These are beyond the scope of this module, but these illustrations should be enough to make you aware that there is a lot to learn here.

List of Figures

Bibliography

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