Mathematics for Physics 2

ODE, Multiple Integrals, Probability Theory

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Contents

Chapter

1 Ordinary Differential Equations

1.1 What is an ODE?

An ordinary differential equation is an equation between an unknown function and its derivatives (of first and higher order), i.e. its solution will be a function and not just a number as for quadratic equations for example.

Let us start with some physical examples:

To start, we fix some notation

An example for a higher order ordinary differential equation is **Newton's law**:

where F is the force on the particle.

A mass m is attached to an elastic spring of force constant k, the other end of which is attached to a fixed point and the spring is supposed to obey Hooke's law with spring constant k.

Then the force

gives the equation of motion

which is a differential equation for the function $x = x(t)$.

1.1.1 Vocabulary

Let us introduce some terms that we will use frequently. We do so by considering the specific example

$$m\frac{d^2x}{dt^2} = -kx. \tag{1.1.1}$$

1. The unknown function x is called the **dependent variable** and t is called the **independent variable**.

2. Equation (1.1.1) is called an **ordinary differential equation** (ODE) because the unknown function depends only on one independent variable.

3. The highest order of derivatives involved in the ODE is called the **order** of the ODE. Equation (1.1.1) is of second order.

4. The ODE (1.1.1) is **linear** because the dependent variable and its derivatives occur only linearly (i.e. in first power).

5. The ODE (1.1.1) has **constant coefficients** since neither m nor k are themselves functions of t. If they were, we would say the equation has **variable coefficients**[1].

6. A given function $x = x(t)$ is a **solution** of the ODE (1.1.1) if the equation is satisfied for all t (possibly in a specified domain).

1.1.2 Types of problems for ODEs

Initial value problems (IVPs)

Claim: The function

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right) \tag{1.1.2}$$

is a solution of

$$m\frac{d^2x}{dt^2} = -kx \tag{1.1.3}$$

for arbitrary constants $A, B \in \mathbb{R}$.

Check:

[1] One then also says that the equation has non-constant coefficients.

We see that equation (1.1.3) has infinitely many solutions. This is a general phenomenon. As you recall from integration of functions, we get a free constant when we integrate, i.e. the integral of a function is not just a function but a set of functions:

$$\int f dx = F + C$$

with $F' = f$.
To solve differential equations, we have to integrate, as we will see later, essentially order of the ODE times which will give us the number of free constants involved in the solution. Since the order of (1.1.1) is two, we have two free constants, in this case called A and B.

How could we fix this? Let us think about a very simple version first. Consider $f(t) = \sin(t) + 3$. Then,

$$F_C(t) = \int f'(t)\mathrm{d}t = \int \cos(t)\mathrm{d}t = \sin(t) + C.$$

What do we need to know to get f back from F_C?

Thus, to determine A and B in (1.1.2), we need to specify the **initial conditions**, i.e. the initial state of the system. Thus, in general, to not get infinitely many solutions, we consider the following problem

$$\begin{cases} \text{Find a solution to the ODE } x = x(t) \\ \text{that satisfies } x(0) = L \\ \text{and } \frac{dx}{dt}(0) = 0. \end{cases} \tag{IVP}$$

These types of problems are called **initial value problem** (IVP).

Let us calculate the particular values of A and B:

Thus, the (IVP) has the **unique** solution

Boundary value problems (BVPs)

Now let us consider a different type of problem. A string of length L is vibrating with angular frequency ω.

The shape of the string, described by $y = y(x)$ at any given time satisfies the ODE

$$c^2 \frac{d^2 y}{dx^2} = -\omega^2 y. \tag{1.1.4}$$

Find the shape under the condition that the string is clamped at the ends, i.e.

In short, we have

$$\begin{cases} \text{Find a solution to the ODE } y = y(x) \\ \text{that satisfies} \\ \text{and} \end{cases} \tag{BVP}$$

which is called a **boundary value problem** (BVP).

The equation (1.1.4) is again a second order equation, it is linear and has constant coefficients. The dependent variable is y and the independent variable is x. We also realise that (1.1.4) is the same as (1.1.3) just with different variable names. Hence, we know the general solution:

For the constants A and B, we obtain

Concluding observations

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- An IVP has 'always' a unique solution if the right number of initial conditions are given.

- A BVP does not always have a solution, it imposes a constraint on the ODE.

1.1.3 Solutions in implicit form

Before we get to this point, let us remind ourselves of some rules of differentiation: Let $y = y(x)$ be a (sufficiently smooth) arbitrary function and find formulas for

$$\frac{d}{dx}\left(x^2 y(x)\right) \quad \text{and} \quad \frac{d}{dx}\left(x^2 \sin\left(y(x)\right)\right).$$

Now, the equation

defines a function $y = y(x)$. (Even though we can not find an explicit formula for it.) This is related to the Lambert W-function. If you are interested, you can have a look at the corresponding Wikipedia article and the resources cited therein.

Show that this function satisfies the ODE

$$\frac{dy}{dx} = \frac{2x}{1+y}e^{-y}. \tag{1.1.5}$$

Answer:

The general solution of the ODE (1.1.5) should contain an arbitrary constant. Indeed,

defines a function $y = y(x)$ for any $C \in \mathbb{R}$. The equation

$$ye^y - x^2 = C$$

gives the solution of (1.1.5) in **implicit form**, i.e. in a form $F(y, x) = 0$ for some F instead of $y = g(x)$ for some g.

Exercise 1.1. *Show that the function y which satisfies the equation*

$$ye^y - x^2 = C$$

satisfies

$$\frac{dy}{dx} = \frac{2x}{1+y}e^{-y}.$$

1.2 Separable ODE

1.2.1 Examples

Example 1.1.

Consider a particle on the number line with position $x(t)$ *and suppose that its velocity* $\frac{dx}{dt}$ *is given by*

$$\frac{dx}{dt} = \frac{3}{(1+t)^2}.$$

Find the particles position $x = x(t)$ *at a time* $t > 0$ *if we suppose that* $x(0) = 0$*.*

Now, we fix the constant:

Remark 1.1. *Ordinary integration is a special case of solving an ODE, where the right hand side does not depend on the function that we seek. The integration constant produces the free constant we need in the general solution.*

Example 1.2.

A body moving slowly through a liquid experiences a friction force that is proportional to its velocity.

a) *A body of mass m is moving through the liquid* **horizontally**. *Its initial velocity is $v(0) = v_0$. Find its velocity as a function of time.*

We may be able to guess a solution to this equation:

We could also rewrite the above differential equation as

A different way of rewriting the same calculation starts with

$$\frac{dv}{dt} = -\frac{\alpha}{m}v$$

transformed to

$$\frac{dv}{v} = -\frac{\alpha}{m}dt \quad \textit{(separation of variables)}$$

which leads with integration to

$$\int \frac{1}{v}dv = \int -\frac{\alpha}{m}dt.$$

After the evaluation of the integrals, we obtain

$$\ln(|v|) = -\frac{\alpha}{m}t + C.$$

Finally, we fix the constant:

b) *The same body is* **falling vertically** *through the liquid. Find its velocity as a function of time if the initial velocity* $v(0)$ *is* v_0*. We have to solve*

$$m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt} + mg$$

or

$$\frac{dv}{dt} = -\frac{\alpha}{m}\left(v - \frac{mg}{\alpha}\right).$$

Now we determine the constant:

$$v_0 = v(0) = \frac{mg}{\alpha} + C$$

which leads to

$$C = v_0 - \frac{mg}{\alpha}.$$

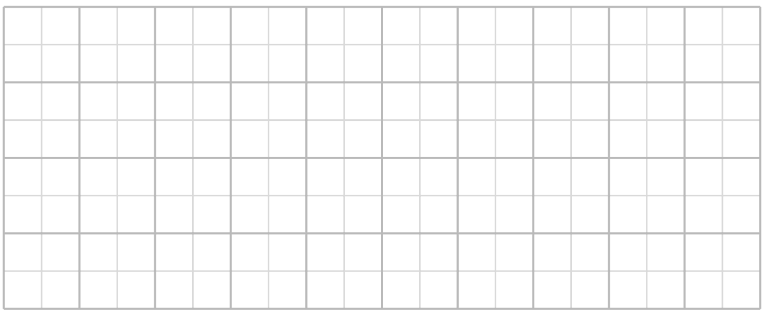
Thus, we get

$$v(t) = \frac{mg}{\alpha} + \left(v_0 - \frac{mg}{\alpha}\right) e^{-\frac{\alpha}{m}t}$$

Discussion:

- *For long time ($t \to +\infty$), we have that*

  at this velocity, $\alpha v = mg$, i.e. friction and the gravitation forces cancel.
- *The asymptotic velocity $\dfrac{mg}{\alpha}$ is approached exponentially with time scale $\dfrac{m}{\alpha}$.*



Example 1.3.

Find the general solution of the ODE

$$\frac{dy}{dx} = \frac{1+x}{2-y}.$$

1.2.2 Summary

A differential equation of the form

$$\begin{cases} \frac{dy}{dt} = f(y)g(t) \\ y(t_0) = y_0 \end{cases} \tag{1.2.1}$$

is called **separable**. If we set

$$F(y) = \int_{y_0}^{y} \frac{1}{f(s)} \mathrm{d}s$$

then, the solution of (1.2.1) is given by

$$y(t) = F^{-1}\left(\int_{t_0}^{t} g(\tau)\mathrm{d}\tau\right). \tag{1.2.2}$$

Exercise 1.2. *Use the rules of differentiation*[2] *to prove that* (1.2.2) *is a solution of* (1.2.1)*. Recall, how one calculates integrals of the type*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,\tau)d\tau.$$

Algorithm in Leibniz's notion

Step 1. Rewrite the ODE

$$\frac{dy}{dt} = f(y)g(t)$$

in separated form, i.e.

$$\frac{dy}{f(y)} = g(t)dt.$$

Step 2. Integrate both sides with respect to the respective variable. Do not forget the integration constant. We only need one!

Step 3. Rearrange for y in terms of t. (That is formula (1.2.2).)

[2] Here that means chain rule and the rule for computing the derivative of the inverse function in terms of the derivative of the original function.

1.3 ODEs that can be transformed in separable form

To introduce this method, let us first look at two examples.

Example 1.4.

Find the general solution of the ODE

$$\frac{dy}{dx} = \frac{3y - 2x}{2x}.$$

This equation is not separable but can be rewritten as as

$$\frac{dy}{dx} = \frac{x\left(3\frac{y}{x} - 2\right)}{2x} = \frac{3}{2}\frac{y}{x} - 1.$$

The right hand side depends only on $\dfrac{y}{x}$*. We make this the new independent variable:*

Example 1.5.

Find the general solution of

$$\frac{dy}{dx} = -\frac{y^2}{xy + x^2} = -\frac{\left(\frac{y}{x}\right)^2}{1 + \frac{y}{x}}.$$

substitute $y = xv(x)$*,*

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

1.3.1 Summary

An ODE of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

is called of **homogeneous degree**.[3] An ODE of homogeneous degree can be solved by transforming it in separable form by the substitution $v = \dfrac{y}{x}$.

[3] Be aware that we will introduce the term homogeneous later to mean something different. Please keep the two notion apart in your mind.

The general solution os obtained as follows: consider

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

and se $v(x) = \dfrac{y}{x}$. Thus, we obtain

$$\frac{dy}{dx} = \frac{y}{x} + x\frac{dv}{dx}.$$

With the original equation, we then obtain

$$f(v) - v = x\frac{dv}{dx}.$$

Thus, we obtain the separated equation

$$\frac{dx}{x} = \frac{dv}{f(v) - v}.$$

From here, one can use the method discussed to solve separable ODE.

1.4 First order linear ODEs

In this section, we want to derive a general solution for the following initial value problem:

$$a(x)\frac{dy}{dx} + b(x)y = f(x).$$

1.4.1 The method of integrating factors

Example 1.6.

The following example is again divided in a couple of steps:

a) *First, we consider a (sufficiently smooth) arbitrary function $y = y(x)$. Find a formula for $\frac{d}{dx}\,(xy(x))$:*

b) *Find the general solution of*

$$x\frac{dy}{dx} + y = e^x.$$

By the above, the equation can be rewritten as *c) Find the general solution of*

$$\frac{1}{x}\frac{dy}{dx} + \frac{y}{x^2} = x^3.$$

This equation can not be rewritten as easily as the equation in b). However, if we multiply the equation by x^2, we can:

Some theory (integrating factor)

- An equation of the form

$$a(x)\frac{dy}{dx} + b(x)y = f(x) \tag{1.4.1}$$

  is called **linear**, i.e. the terms involving y and $\dfrac{dy}{dx}$ are all in first power.

- If $b(x) = \dfrac{da}{dx}(x)$, then (1.4.1) is

  which can be rewritten as

This equation can be integrated and gives

$$a(x)y = \int f(x)\mathrm{d}x.$$

- If b is not the derivative of a, then equation (1.4.1) can be multiplied by a function $\mu(x)$ such that it can be integrated. First, we divide by $a(x)$ and obtain

---

where

$$p(x) = \frac{b(x)}{a(x)}, \quad q(x) = \frac{f(x)}{a(x)}.$$

If we multiply an ansatz function $\mu = \mu(x)$ and obtain

When we ask

$$\frac{d\mu}{dx} = \mu p(x) \quad \Rightarrow \quad \mu(x) = \exp\left(\int p(x)\mathrm{d}x\right).$$

With that, we can rewrite

$$\frac{dy}{dx} + p(x)y = q(x)$$

to

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)q(x).$$

Hence

$$\mu(x)y = \int \mu(x)q(x)\mathrm{d}x$$

and finally

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\mathrm{d}x.$$

- With that we have an explicit solution of (1.4.1). In calculating the integral for μ, we do not need the integration constant. We only need one function μ that transforms (1.4.1) to integrable form.

The function μ is called an **integrating factor**.

Exercise 1.3. *Find the general solution of*

$$\frac{dy}{dx} + \frac{2}{x}y = x^2.$$

Example 1.7.

Assume the capacitor is initially charged to charge Q_0. At time $t = 0$, the switch is closed so that the capacitor is connected to a voltage source U via a resistor R. Find the charge $Q(t)$ on the capacitor as a function of time.

Physics:

- *The voltage across the resistor and the capacitor must add to U.*
- *The voltage across the resistor is RI, where I is the current through the resistor.*
- *The voltage across the capacitor is $\frac{Q}{C}$.*
- *The currents through the resistor and the capacitor must be the same.*
- *If in time Δt a charge ΔQ is added to the capacitor, the current is $I = \frac{\Delta Q}{\Delta t}$. If $\Delta t \to 0$, we obtain $I = \frac{dQ}{dt}$.*

We translate that into equations:

This can be rewritten to

To solve the equation, we calculate the integrating factor

Thus, we obtain

Fix constant k by initial value:

In particular, if we start with an uncharged capacitor $Q_0 = 0$, then

Further,

$$\begin{aligned} I(t) &= \frac{dQ}{dt} = CU\left(0 - \frac{1}{-RC}\exp\left(-\frac{t}{RC}\right)\right) \\ &= \frac{U}{R}\exp\left(-\frac{t}{RC}\right) \end{aligned}$$

decays exponentially to zero.

Exercise 1.4. *Confirm the above derivative* $\dfrac{dQ}{dt}$.

Summary of the method of integrating factor

As we have seen above, an **integrating factor** is a functions with which we can multiply a differential equation on order to transform it to an integrable equation.
Given an equations of the form

$$\frac{dy}{dx} + p(x)y = q(x),$$

we get

$$\mu(x)\left(\frac{dy}{dx} + p(x)y\right) = \frac{d}{dx}\left(\mu(x)y\right) = \mu(x)q(x)$$

if we set

$$\mu(x) = \exp\left(\int p(x)\mathrm{d}x\right).$$

With that, the general solution is

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\mathrm{d}x.$$

The integration constant of the integral on the right hand side is the free constant of the general solution that can be fixed by an initial condition.

Example 1.8. *To solve*

$$\frac{dy}{dx} + x^2y = e^x,$$

we calculate

$$\mu(x) = \int x^2\mathrm{d}x = \frac{1}{3}x^3.$$

Then, we get

$$\underbrace{e^{\frac{1}{3}x^3}\frac{dy}{dx} + e^{\frac{1}{3}x^3}x^2y}_{=\frac{d}{dx}\left(e^{\frac{1}{3}x^3}y\right)} = e^{\frac{1}{3}x^3+x}.$$

Hence, we get

$$y(x) = e^{-\frac{1}{3}x^3}\int e^{\frac{1}{3}x^3+x}dx.$$

1.4.2 The method of variation of parameters

In this section, we consider the general linear first order ODE

$$\frac{dy}{dx} = a(x)y + b(x) \tag{1.4.2}$$

with initial datum $y(x_0) = y_0$, where x_0 and y_0 are real numbers.

Some more vocabulary:

- We can see that the equation is **linear**. See page 3.
- The equation above is said to be **inhomogeneous** if b is not identically 0 and **homogeneous** if $b(x) = 0$ for all x.

  Note that the term does not involve the dependent variable.

We can solve the homogeneous equation equation

$$\frac{dy}{dx} = a(x)y \tag{1.4.3}$$

by separation of variables:

$$\underbrace{\int \frac{1}{y}\frac{dy}{dx}\mathrm{d}x}_{=\ln|y|} = \int a(x)\mathrm{d}x = \int_{x_0}^{x} a(u)\mathrm{d}u + C \tag{1.4.4}$$

and, thus, we get

$$y_h(x) = K\exp\left(\int_{x_0}^{x} a(u)\mathrm{d}u\right). \tag{1.4.5}$$

The particular limits of the integral over a are chosen to pick the integration constant $C = 0$ as we have the needed integration constant K in front of (1.4.5). We allow K to be an arbitrary number to resolve the modulus $|y|$ in the result of the integral of the left hand side in (1.4.4). We use the index h to indicate that the y_h is a solution of the homogeneous equation associated to (1.4.2)

This is, however, only part of the solution as we want to solve (1.4.2) and not just (1.4.3).

The next step is called **variation of parameters**. We set $K = K(x)$ and make the ansatz

$$y_p(x) = K(x)\exp\left(\int_{x_0}^{x} a(u)\mathsf{d}u\right).$$

Now, we calculate the derivative $\frac{dy_p}{dx}$:

$$\begin{aligned}\frac{dy_p}{dx} &= \frac{dK}{dx}\exp\left(\int_{x_0}^{x} a(u)\mathsf{d}u\right) \\ &\qquad + K(x)a(x)\exp\left(\int_{x_0}^{x} a(u)\mathsf{d}u\right) \\ &= \frac{dK}{dx}\exp\left(\int_{x_0}^{x} a(u)\mathsf{d}u\right) + a(x)y_p(x).\end{aligned}$$

From that equation, we obtain that

$$\frac{dK}{dx}\exp\left(\int_{x_0}^{x} a(u)\mathsf{d}u\right) = b(x)$$

which gives

$$K(x) = \int_{x_0}^{x} b(z)e^{-\int_{x_0}^{z} a(u)\mathsf{d}u}\mathsf{d}z.$$

We can drop[4] the integration constant as we only need a particular $K = K(x)$. This is equivalent to choosing the particular definite integral

$$K(x) = \int_{x_0}^{x} b(z)e^{-\int_{x_0}^{z} a(u)\mathsf{d}u}\mathsf{d}z.$$

With that, the final solution is the superposition

$$\begin{aligned}y(x) &= y_h(x) + y_p(x) \\ &= \left(K + \int_{x_0}^{x} b(z)e^{-\int_{x_0}^{z} a(u)\mathsf{d}u}dz\right)\exp\left(\int_{x_0}^{x} a(u)\mathsf{d}u\right).\end{aligned}$$

As before, one can determine the constant K if an initial datum is given.

[4] i.e. choose it to be zero

With that, we have proven the following theorem:

Theorem 1.1.

Consider

$$\begin{cases} \frac{dy}{dx} = a(x)y + b(x) \\ y(x_0) = y_0 \end{cases} . \tag{IVP}$$

Then,

$$y(x) = \left(y_0 + \int_{x_0}^{x} b(z) e^{-\int_{x_0}^{z} a(u) du} dz\right) \exp\left(\int_{x_0}^{x} a(u) du\right)$$

solves the IVP.

Exercise 1.5. *Solve the inhomogeneous IVP*

$$\begin{cases} \frac{dy}{dx} = \sin(x)y + \\ y(0) = 1 \end{cases}$$

with the method of variation of constants.

Exercise 1.6. *Solve the non-homogeneous IVP*

$$\begin{cases} \frac{dy}{dx} = xy + e^{3x}(x-3) \\ y(0) = 1 \end{cases}$$

with the method of variation of constants.

1.5 Second order linear ODEs

A linear second order ODE has the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x) \tag{1.5.1}$$

with known functions a, b, and c which are called the coefficients and a known function f which is called the right hand side, **forcing function**.

Vocabulary:

1. The equation is of **second order**[5]. See also Page 2.
2. The equation is **linear**. See also Page 3.
3. If the functions a, b, and c are constant, we say that the differential equation has **constant coefficients**. If the a, b, or c are variable functions, we say the equation has **variable coefficients**.
4. If $f(x) = 0$ for all x, then the equation is called **homogeneous**. If f is not identically 0, we call the equation **non-homogeneous**. Pay attention that this is different from homogeneous in Section 1.3.
   Sometimes, the homogeneous equation associated to a non-homogeneous equation is called reduced equation:
   $$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \tag{1.5.2}$$
5. **Superposition** of solutions. This is a general property of linear equations. If y_1 and y_2 are solutions to (1.5.2), then $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution for any constants α_1 and α_2.

[5] if a is not identically 0.

1.5.1 The superposition principle

We have the following result which is making more precise the situation on pages 3 and 5. As you can easily check, the functions

$$\cos\left(\sqrt{\frac{k}{m}}t\right) \quad \text{and} \quad \sin\left(\sqrt{\frac{k}{m}}t\right)$$

are solutions to

$$m\frac{d^2x}{dt^2} = -kx. \tag{1.5.3}$$

As we have shown in Section 1.1.2, the function

$$\alpha_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + \alpha_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

is a solution to (1.5.3) for arbitrary constants α_1 and α_2.

Theorem 1.2 (Superposition of solutions of linear 2nd order equations)**.**
Let y_1 and y_2 be solutions of (1.5.2). Then, for all constants α_1 and α_2, the function $\alpha_1 y_1 + \alpha_2 y_2$ is a solution of (1.5.2).

Let us check:

1.6 Solution of second order linear equations

If y_h is a solution of the homogeneous equation

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \tag{1.6.1}$$

and y_p a solution of

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x),$$

then

$$y(x) = y_h(x) + y_p(x)$$

is a solution of (1.5.1).

Exercise 1.7.

Check the last claim by a similar calculation as in Section 1.5.1.

To solve the non-homogeneous equation (1.5.1):

1) Find **two**[6] solutions y_1 and y_2 of the homogeneous equation (1.6.1). Then

$$y_h(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x)$$

is the general solution of the homogeneous equation (1.6.1). This solution y_h is called **complementary function**.[7]

2) Find **one** particular solution y_p of the full equation (1.5.1). Then

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= \alpha_1 y_1(x) + \alpha_2 y_2(x) + y_p(x) \end{aligned}$$

is the general solution of (1.5.1).

The function y_p is called a **particular integral** or a **particular solution** of (1.5.1).

1.6.1 Constant coefficients

We consider the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x). \tag{1.6.2}$$

To find the two solutions of the homogeneous equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

associated to (1.6.2), we write down the **auxiliary equation**[8]

$$a\lambda^2 + b\lambda + c = 0$$

by replacing $\dfrac{d^k y}{dx^k}$ with λ^k.

[6] The two is coming from second order. For equations of higher order, you have to find order many solutions.

[7] **Complementary function**: the part of the general solution of a linear differential equation which is the **general solution of the associated homogeneous equation** obtained by substituting zero for the terms not containing the dependent variable.

[8] We also call this equation the characteristic polynomial of the differential equation.

The auxiliary equation is coming from the ansatz $y(x) = e^{\lambda x}$ substituted into the equation:

The classification of quadratics

Example 1.9.

Find the general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 0.$$

Example 1.10.

Find the general solution of

$$2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = 0.$$

Example 1.11.

Find the general solution of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

Summary

To solve the homogeneous second-order linear ODE with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

we first solve the auxiliary equation

$$a\lambda^2 + b\lambda + c = 0. \tag{1.6.3}$$

We have

$$\lambda_\pm = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{a^2} - \frac{c}{a}} = -\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

	Solutions of (1.6.3)	General solution of ODE
a)	two distinct real solutions λ_- and λ_+	$y(x) = \alpha_1 e^{\lambda_- x} + \alpha_2 e^{\lambda_+ x}$
b)	two complex conjugated solutions $\lambda_\pm = \alpha \pm i\beta$	$y(x) = e^{\alpha}(\alpha_1 \cos(\beta x) + \alpha_2 \sin(\beta x))$
c)	one double solution $\lambda = \lambda_\pm$	$y(x) = (\alpha_1 x + \alpha_2)e^{\lambda x}$

1.6.2 Examples

Example 1.12.

A mass m is attached to an elastic spring of force constant k, the other end of which is attached to a fixed point. The spring is supposed to obey Hooke's law, namely that, when it is extended (or compressed) by a distance x from its natural length, the tension (or thrust) in the spring is kx, and the equation of motion is

$$m\frac{d^2x}{dt^2} = -kx$$

or

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \quad \omega_0^2 = \frac{k}{m}$$

a linear, second order, homogeneous ODE with constant coefficients.

The auxiliary equation is

The solutions are

Hence, the general solution is

This can be written as

Relating the two forms

The form shows that the solution is a harmonic oscillation with amplitude a, frequency ω_0.

Example 1.13. *Now, we consider the same situation as in Example 1.12 but introduce friction as a more physical assumption.*
We assume that the friction force is proportional to the velocity of the particle. Thus, we get the equation

This is a second order linear ODE with constant coefficients.

1.6.3 Finding particular solutions

There is a general method to find particular solutions which we have already discussed in the case of first order linear equations: **variation of parameters**. See also Kreyszig 2.10. You can also find the method described for second order equations in Paul's Online Math Notes.

In many physically relevant cases, we can guess a particular solution. The general idea is as follows: The particular solution should have the 'same form' as the inhomogeneous ('driving') term.

We illustrate that with an example:

Example 1.14.

Find the general solution of

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2x$$

Step 1: We find the complementary function, i.e. the solution y_h to the homogeneous problem.

Step 2: Finding a particular integral. The driving term is a

Thus, we make the ansatz

Now we determine the parameters:

Thus, the general solution is

Example 1.15.

Find the solution of

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 5e^{3x}$$

with the initial conditions

$$y(0) = -1, \quad y'(0) = 0.$$

Step 1: We find the complementary function, i.e. the solution y_h to the homogeneous problem.

Step 2: Finding a particular integral. The driving term is a

Thus, we make the ansatz

Now we determine the parameters:

Thus, the general solution is

Step 3: Finally, we determine the constants:

Example 1.16.

Find the general solution of

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^{-2x}$$

with the initial conditions

$$y(0) = \frac{9}{2}, \quad y'(0) = 1.$$

Step 1: The complementary function is

Step 2: Finding a particular integral. The driving term is a

The usual ansatz would be

$$y_p(x) = Re^{-2x}.$$

However, that is part of the complementary function. If we substitute this ansatz into the ODE, we will get zero:

We modify the usual ansatz:

Step 3: Fially, we determine the constants:

Example 1.17.

In Example 1.12 on Page 43, we have discussed the dynamics of a damped oscillator. Now, we want to study the same oscillator when an external force

$$F_{ext} = F_0 \cos(\Omega t)$$

is acting on it.[9]

The equation of motion is

A bit shorter

with

Let us find the general solution to this equation:

1. *The complementary function id the general solution of*

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = f_0 \cos(\Omega t).$$

   We know the solution from Example 1.12. We must distinguish the cases of weak, critical, or strong damping.

2. *We find a particular integral. We make the ansatz*

[9] A discussion of this example can be found here in the lecture notes on *Oscillations and waves* from Richard Fitzpatrick at the University of Texas at Austin.

Now we discuss the amplitude $a(\Omega)$ of the steady state oscillator as a function of the driving frequency Ω:

$$a(\Omega = 0) = \frac{f_0}{\omega_0^2} \neq 0.$$

For large Ω:

$$a(\Omega) = \frac{f_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Omega^2\gamma^2}} \approx \frac{f_0}{\sqrt{\Omega^4 + \Omega^2\gamma^2}} \approx \frac{f_0}{\Omega^2}.$$

Are there maxima/minima?

Chapter

2 Multiple Integrals

Multiple integrals are integrals of functions depending on more than one variable. They are useful to calculate

- masses,
- volumes,
- centres of mass,
- moments of inertia,

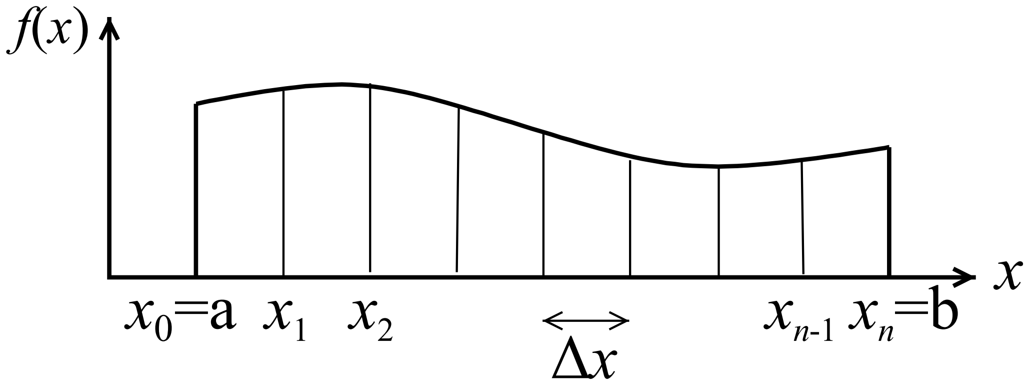
and many other quantities. We will start by looking at 'double integrals', or integrals over areas and then look at 'triple integrals'.

2.1 Ordinary Integration

As you have learned,

$$\int_a^b f(x)\mathrm{d}x$$

measures the (signed) area under the graph of $y = f(x)$.



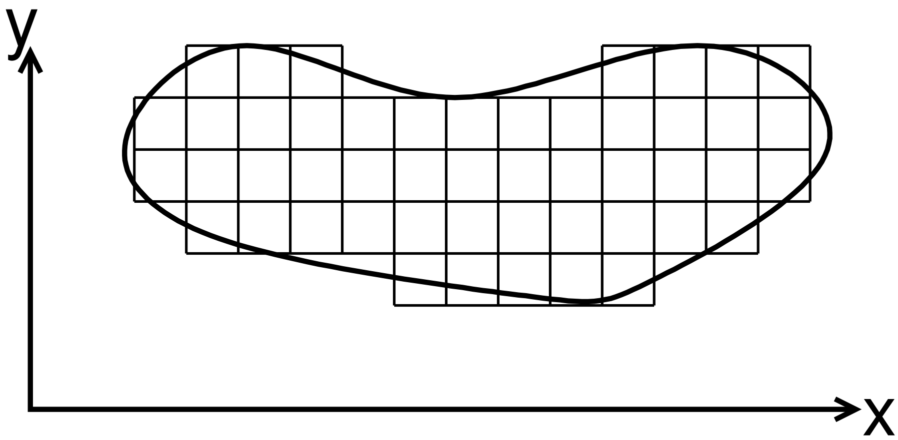
Divide the length $[a, b]$ into intervals of width Δx.

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= \lim_{\Delta x \to 0} \sum_{\text{intervals}} \text{width of interval} \times \text{height} \\ &= \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x\end{aligned}$$

2.2 Integrals over areas

Suppose that we want to calculate the volume under a graph of $f(x, y)$ over some region A in the xy-plane.

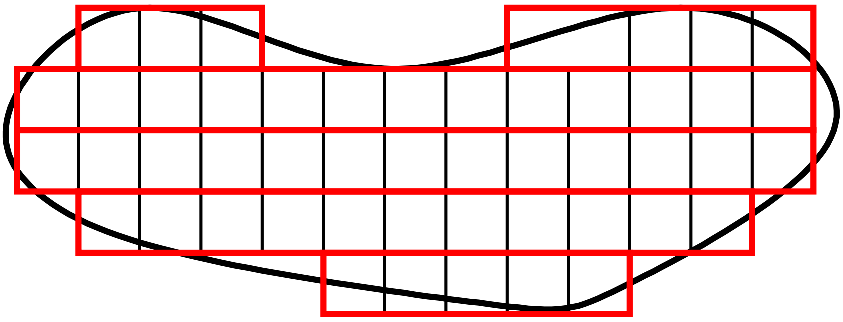
To do this, first divide A up into rectangles with dimensions $\Delta x, \Delta y$.



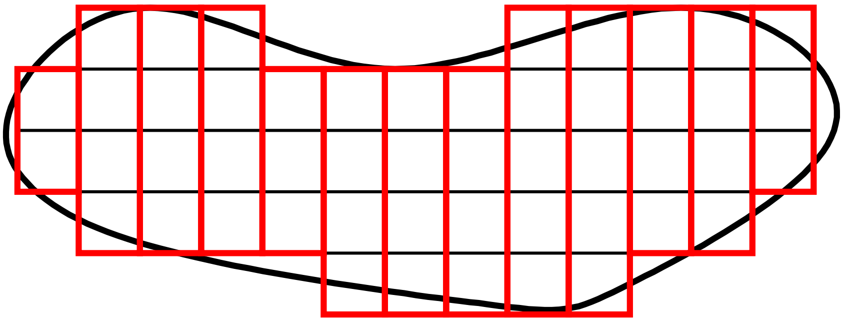
$$\text{Volume under graph} = \lim_{\Delta x, \Delta y \to 0} \sum_{\text{rectangles}} \text{area of rectangle} \times \text{height}.$$

There are two ways to order the sum:

(i) First add up the volumes of all the cuboids in each row, then sum over rows.



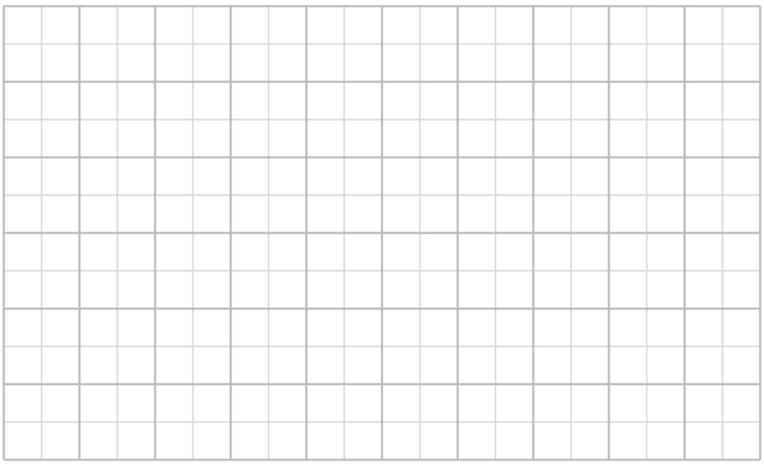
(ii) First add up the volumes of all the cuboids in each column, then sum over columns.



2.2.1 Integrals over rectangles

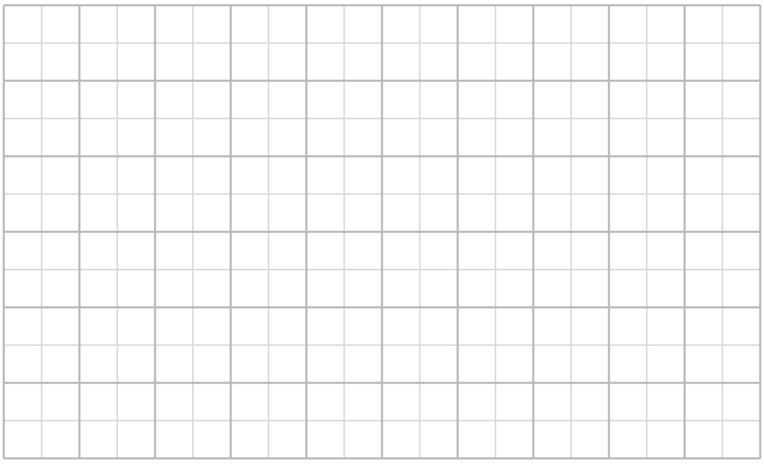
Assume that A is a rectangle defined by $a \leq x \leq b$ and $c \leq y \leq d$.

(i) sum over rows first:



$$
\begin{aligned}
\text{Volume under graph} &= \lim_{\Delta x, \Delta y \to 0} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} f(x_i, y_j) \Delta x \Delta y \right) \\
&= \lim_{\Delta y \to 0} \sum_{j=1}^{n} \left(\int_a^b f(x, y_j) \mathsf{d}x \right) \Delta y \\
&= \int_c^d \left(\int_a^b f(x, y) \mathsf{d}x \right) \mathsf{d}y
\end{aligned}
$$

(ii) sum over columns first:



$$
\begin{aligned}
\text{Volume under graph} &= \lim_{\Delta x, \Delta y \to 0} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} f(x_i, y_j) \Delta x \Delta y \right) \\
&= \lim_{\Delta x \to 0} \sum_{i=1}^{m} \left(\int_c^d f(x_i, y) \mathsf{d}y \right) \Delta x \\
&= \int_a^b \left(\int_c^d f(x, y) \mathsf{d}y \right) \mathsf{d}x
\end{aligned}
$$

Example 1

Example 2

Exercise 2.1. *Verify the following integrals:*

$$\begin{aligned}
&\int_{-1}^{2}\int_{-1}^{2} x^2 y \, dy \, dx = \frac{9}{2},\\
&\int_{-1}^{2}\int_{-2}^{3} x\sin(\pi y) \, dy \, dx = \frac{3}{\pi},\\
&\int_{0}^{4}\int_{0}^{1} xy\sin(\pi y) \, dy \, dx = \frac{8}{\pi},\\
&\int_{0}^{1}\int_{0}^{1} \sin(\pi y) x^2 e^x \, dy \, dx = \frac{2(e-2)}{\pi}, and\\
&\int_{\pi}^{2\pi}\int_{\pi}^{2\pi} e^y \sin(x)\cos(y) \, dy \, dx = -e^{\pi}(1+e^{\pi}).
\end{aligned}$$

2.2.2 Order of integration (rectangular regions)

In practice, we do not need to write the brackets in the expression

$$\int_c^d \left(\int_a^b f(x,y) \mathrm{d}x \right) \mathrm{d}y.$$

If we write

$$\int_c^d \int_a^b f(x,y) \mathrm{d}x \, \mathrm{d}y,$$

we understand that this means

- first integrate with respect to x from a to b,
- then integrate with respect to y from c to d.

If we write

$$\int_a^b \int_c^d f(x,y) \mathrm{d}y \, \mathrm{d}x,$$

we understand that this means

- first integrate with respect to y from c to d,
- then integrate with respect to x from a to b.

i.e. you work from the inside to the outside.

For **rectangular regions** both integrals give the same answer.

2.2.3 Order of integration (curved regions)

In general, the integral of $f(x,y)$ over a non-rectangular region A could be written as

$$\iint_A f(x,y) \mathrm{d}x \, \mathrm{d}y = \int_c^d \int_{a(y)}^{b(y)} f(x,y) \mathrm{d}x \, \mathrm{d}y$$

or

$$\iint_A f(x,y) \mathrm{d}y \, \mathrm{d}x = \int_p^q \int_{r(x)}^{s(x)} f(x,y) \mathrm{d}y \, \mathrm{d}x.$$

The limits $a(y), b(y), c, d$ will differ from the limits $p, q, r(x), s(x)$.

In general, we have

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\mathrm{d}x\,\mathrm{d}y \neq \int_{a(y)}^{b(y)} \int_c^d f(x,y)\mathrm{d}y\,\mathrm{d}x$$

and

$$\int_p^q \int_{r(x)}^{s(x)} f(x,y)\mathrm{d}y\,\mathrm{d}x = \int_{r(x)}^{s(x)} \int_p^q f(x,y)ddx\,\mathrm{d}y$$

If the function $f(x,y)$ is continuous over A, one can, just as in the case of rectangular regions, interchange the order of integration. However, as already noted, one has to adjust the limits in the integrals.

Exercise 2.2. *Use double integrals to compute the area of a circle of radius $r > 0$, i.e. compute*

$$\int\limits_{-r}^{r} \int\limits_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} 1\,dxdy.$$

Write down how the integral would look like if you exchange the order of integration. Why are the limits what they are?

Example 1: curved boundary

Example 2: curved boundary

Example: Splitting of regions

Sometimes it is necessary to split the region of integration into smaller pieces.

2.3 Triple integrals

In the last section, we have learned how

- double integrals are defined,
- double integrals over rectangular regions are calculated, and
- double integrals over curved regions are calculated.

Now, we will do the same for triple integrals.

2.3.1 Some motivation

Suppose that we know the mass density $\rho(x, y, z)$ of an object. How do we find its mass?

Solution: We divide the object into small boxes of dimensions Δx, Δy, Δz, find the mass of each box, which is approximately

$$\rho(x_i, y_j, z_k)\Delta x\Delta y\Delta z$$

and then add all the masses.

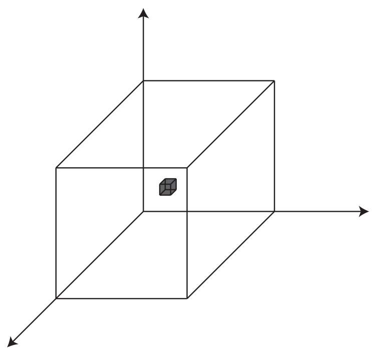


Figure 2.1: A small cuboid mass in a bigger object.

Taking the limit results in a triple integral

$$\text{Mass} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \sum_{\text{all boxes}} \rho(x_i, y_j, z_k)\Delta x\Delta y\Delta z.$$

2.3.2 Evaluating triple integrals

Triple integrals are of the form

$$\int_a^b \int_{H_1(x)}^{H_2(x)} \int_{G_1(x,y)}^{G_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

We can justify performing the integrals sequentially by considering the figures below:

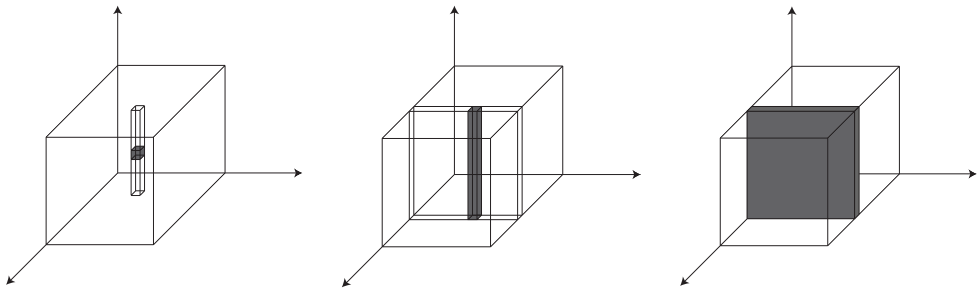


Figure 2.2: Integrating over a column (a), a slab (b) and finally the whole volume (c).

2.3.3 Examples

2.4 Some applications of multiple integrals

In the previous sections, we learned about double and triple integrals.

In the next section, we will see how to use these to calculate:

- the masses, areas, and volumes
- the centres of mass
- moments of inertia
- forces due to pressure

Figure 2.3: "Hang on lads, I've got a great idea..." by the artist Richard Wilson. At the De La Warr Pavilion in Bexhill.

2.4.1 Some notation

It is common to write

$$\iint_A f \, \mathrm{d}A = \iint_A f(x,y) \, \mathrm{d}A$$

as a shorthand for the double integral of a function $f(x,y)$, where A denotes the region of integration. Similarly,

$$\iiint_V g \, \mathrm{d}V = \iiint_V g(x,y,z) \, \mathrm{d}V$$

means the integral of a function $g(x,y,z)$ over a volume V.

2.4.2 Mass, area, and volume

Suppose that a 2D object is described by a region A in the plane with mass per unit area $\lambda(x,y)$. Then

- the area of the object is $\displaystyle\iint_A 1 \, \mathrm{d}A$;
- the mass of the object is $\displaystyle\iint_A \lambda \, \mathrm{d}A$.

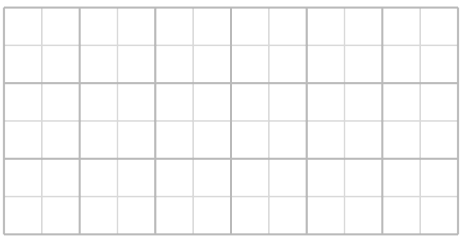
Similarly, for a 3D object described by a volume V with mass per unit volume $\rho(x,y,z)$,

- the volume of the object is $\displaystyle\iint_V 1 \, \mathrm{d}V$;
- the mass of the object is $\displaystyle\iint_V \rho \, \mathrm{d}V$.

2.4.3 Centre of mass

The centre of mass of a 2D object is a point (x_0, y_0). It is calculated using the formulae:

$$x_0 = \frac{1}{\text{mass}} \iint_A x\lambda(x,y) \, \mathrm{d}A, \qquad y_0 = \frac{1}{\text{mass}} \iint_A y\lambda(x,y) \, \mathrm{d}A.$$



The centre of mass (x_0, y_0, z_0) of a 3D object is given by

$$\begin{aligned} x_0 &= \frac{1}{\text{mass}} \iiint_V x\rho(x,y,z)\mathrm{d}V, \\ y_0 &= \frac{1}{\text{mass}} \iiint_V y\rho(x,y,z)\mathrm{d}V, \\ z_0 &= \frac{1}{\text{mass}} \iiint_V z\rho(x,y,z)\mathrm{d}V. \end{aligned}$$

2.4.4 Moment of inertia

The *moment of inertia* of an object measures how difficult it is to rotate.

Just as objects with large masses are difficult to move, objects with large moments of inertia are difficult to rotate.

The *angular momentum* L of a body with moment of inertia I rotating at angular velocity ω is given by

$$L = I\omega.$$

Angular momentum is conserved if no external forces act on the body.

Calculating moments of inertia

The moment of inertia of a 3D object rotating about the x-axis is given by

$$I_x = \iiint_V (y^2 + z^2)\rho(x,y,z)\mathrm{d}V.$$

Similarly, the moments of inertia about the y- and z-axes are

$$\begin{aligned} I_y &= \iiint_V (x^2 + z^2)\rho(x,y,z)\mathrm{d}V \\ I_z &= \iiint_V (x^2 + y^2)\rho(x,y,z)\mathrm{d}V. \end{aligned}$$

For 2D objects:

$$I_x = \iint_A y^2\lambda\mathrm{d}A, \quad I_y = \iint_A x^2\lambda\mathrm{d}A, \quad I_z = \iint_A (x^2 + y^2)\lambda\mathrm{d}A.$$

2.4.5 Example

2.4.6 Forces due to pressure

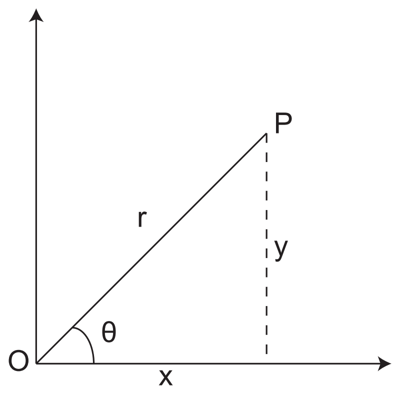
The force due to a pressure $P(x, y)$ on an area A is

$$\iint_A P(x, y)\,\mathrm{d}A.$$

Figure 2.4: Flaming Gorge Dam, Utah

2.5 Integrals in polar coordinates

We have



Using basic trigonometry, we obtain

- Given (r, θ), then (x, y) are given by

$$
\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta)
\end{aligned}
$$

- Given (x, y), then (r, θ) are given by

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\tan(\theta) &= \frac{y}{x}
\end{aligned}
$$

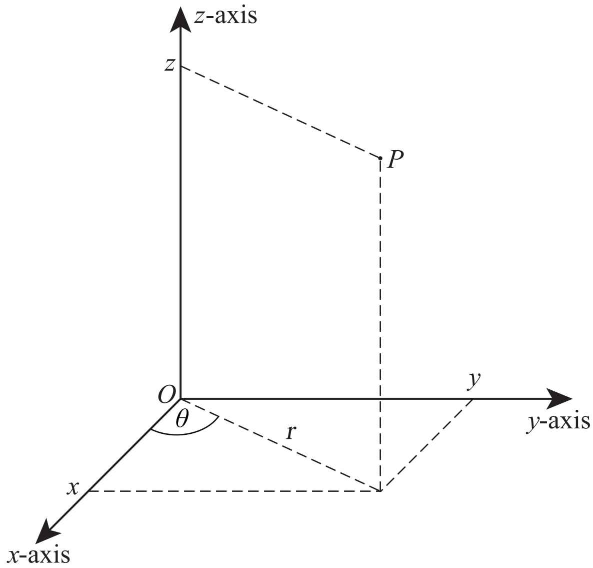
We use the following formula to calculate integrals in polar coordinates:

$$\iint_A f \mathsf{d}A = \iint_A f(r,\theta) r \mathsf{d}r \mathsf{d}\theta.$$

2.5.1 Examples

2.6 3D cylindrical polar coordinates

We have



Again, using elementary trigonometry, one gets

$$
\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
z &= z
\end{aligned}
$$

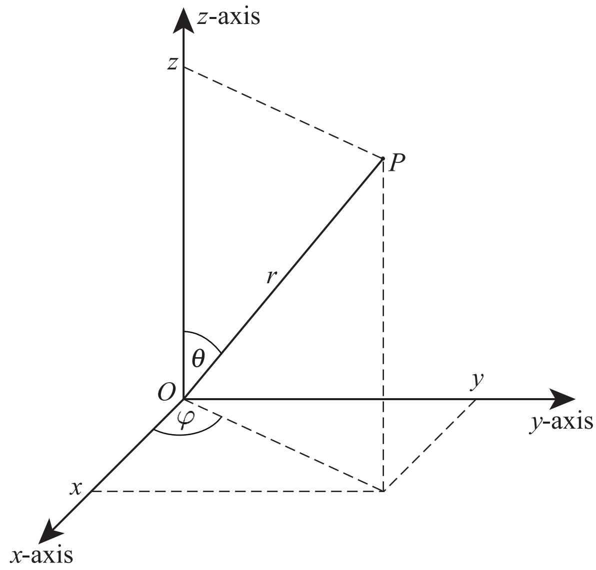
We use the following formula to calculate integrals in cylindrical coordinates:

$$\iiint_V f \mathsf{d}V = \iiint_V f(r,\theta,z) r \mathsf{d}r \mathsf{d}\theta \mathsf{d}z.$$

2.6.1 Examples

2.7 3D spherical coordinates

We have



Again, using elementary trigonometry, one gets

$$\begin{aligned} x &= r\sin(\theta)\cos(\varphi) \\ y &= r\sin(\theta)\sin(\varphi) \\ z &= r\cos(\theta) \end{aligned}$$

We use the following formula to calculate integrals in cylindrical coordinates:

$$\iiint_V f\mathsf{d}V = \iiint_V f(r,\theta,\varphi)r^2\sin\theta\mathsf{d}r\mathsf{d}\theta\mathsf{d}\varphi.$$

2.7.1 Examples

Chapter

3 Probability Theory

Consider an experiment with random outcome and we are going to assume that the experiment is repeated many times.

The number of times a given outcome is obtained is called its **absolute frequency**. The absolute frequency divided by the total number of repetitions is called the **relative frequency**. The relative frequency is a number between 0 and 1.

Example 3.1. *Die with data table.*

If an experiment is repeated many times, the relative frequencies will stabilise at certain values.[1] These are called the **probabilities** of the outcomes. Since the relative frequencies are numbers between 0 and 1, the probabilities are between 0 and 1. An outcome with probability 0 is **impossible**. An outcome with probability 1 is **certain**. The probabilities of all outcomes have the sum 1.

A collection of some or all outcomes of an experiment is called an **event**. The collection of all possible outcomes is usually defined by Ω. An event A is a sub-set of Ω.

The probability $P(A)$ is calculated as

$$P(A) = \frac{\#(A)}{\#(\Omega)} = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}},$$

where the function $\#(S)$ counts the number of elements in the set S. This definition only works if Ω is finite.

The set of all possible events is the power set 2^{Ω} of Ω, i.e. the set of all sub-sets of Ω.

[1] Strictly speaking that is not true and probability should be treated axiomatically following the approach of Kolmogorov. However, this approach is mathematically more difficult.

3.0.1 Elementary probability theory

Example 3.2 (Throwing a die)**.**

For a die, the set Ω is given by $\{1, 2, 3, 4, 5, 6\}$.

- *The probability to throw a 4 is $\frac{1}{6}$.*
- *The probability to throw an even number is $\frac{3}{6} = \frac{1}{2}$.*
- *The probability to throw at least 3 is $\frac{4}{6} = \frac{2}{3}$.*

Example 3.3 (Throwing two dice)**.**

We consider pair of perfectly fair dice.

1. *What is the probability that the total score is 5? The total score can be between 2 and 12, but these results are not equally likely. If we denote an outcome by*

$$(\text{first die score}, \text{second die score}),$$

then the 36 outcomes

$$\begin{matrix} (1,1) & (1,2) & \dots & (1,6) \\ (2,1) & \dots & \dots & (2,6) \\ \vdots & & & \vdots \\ (6,1) & \dots & \dots & (6,6) \end{matrix}$$

are equally likely. There are four outcomes with total score 5:

Hence,

---

2. *What is the probability that the total score is 6?*

---

3. *What is the probability that the total score is not 5? We can count the outcomes with scores other than 5. However, it is simpler to calculate*

4. *What is the probability that a double is thrown, i.e. the numbers on both dice are equal?*

5. *What is the probability that a double is thrown or the total score is 5?*

   a) *Combine the favourable outcomes of $1.$ and $4.$:*

   b) *We alternatively compute:*

6. *What is the probability that a double is thrown or the total score is 6? We combine the favourable outcomes from $2.$ and $4.$:*

Let us collect some of the rules we have used so far and introduce some notation:

The empty event is denoted by $\emptyset$. given two events A and B, we denote the event 'A and B' by $P \cap A$ and the event 'A or B' by $A \cup B$.

1. We have the trivial probabilities $P(\emptyset) = 0$ and $P(\Omega) = 1$.
2. For every event A, we have

$$P(A) + P(A^c) = 1,$$

where A^c is the complement event of A, i.e. 'not A'.

3. If events A, B are mutually exclusive, in symbols $A \cap B = \emptyset$. Then

$$P(A \cup B) = P(A) + P(B).$$

4. For any events A and B, we have

$$P(A \cup B) = P(A) + P(B) + P(A \cap B).$$

Remark 3.1.

Case 3. is a special case of 4. If A and B are mutually exclusive, then $P(A \cap B) = 0$. The case 2. is a special case of 2.:

$$P(A) + P(A^c) = P(\underbrace{A \cup A^c}_{=\Omega}) = 1$$

since $A \cap A^c = \emptyset$.

Example 3.4 (Throwing two dice continued)**.**

7. *What is the probability that the first die shows at least 5 and the second die shows an even number. Listing all the possibilities:*

   Or

   Note: This argument only works because the two events are **independent***, i.e. the result on die 1 does not influence the result on die 2.*

8. *What is the probability to have at least 5 on the first die and a total score of at least 8?*

   Possibilities:

   Note:

Example 3.5. *You throw a die and a coin. What is the probability that you throw a 6 and 'heads'?*

Alternative argument:

Calculating the probability of 'A and B', $P(A \cap B)$, for two independent events A and B is calculated by

$$P(A \cap P) = P(A)P(B).$$

3.1 Excursus: Combinatorics - How to count

Example 3.6. *To introduce the different counting numbers as factorial and the binomial coefficient, we will use illustrative examples.*

1. *I want to fly 10 different flags on 10 poles. How many possibilities do I have to arrange them?*

   Rule: *There are* $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$ *(speak* n *factorial) permutations (i.e. different orderings) of* n *objects.*

2. *The flags are not all different. I have seven blue and three yellow ones. In how many ways can I arrange them?*
   As we have seen before, there are $10!$ *arrangements of the flags in total. Since it does not matter if we rearrange the blue or yellow ones within themselves, we need to remove the number of arrangement for each.*
   There are $7!$ *ways to rearrange the blue ones and* $2!$ *ways to arrange the yellow ones. Thus, there are*

   possibilities.

Different interpretations:

There are 102 possibilities to pick 7 out of 10 poles on which to fly the blue flags.

Rule: *There are*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

ways to choose k objects out of a collection of n. This number is called binomial coefficient*.*

3. *In how many ways can I arrange 5 blue, 3 yellow, and 2 red flags?*

3.2 Mean and variance

I throw a die 100 times and obtain

	1	2	3	4	5	6
abs. freq.	21	14	22	12	16	15
rel. freq.	0.21	0.14	0.22	0.12	0.16	0.15

The mean of the result is

$$\begin{aligned}\overline{X} &= \frac{21\cdot 1+14\cdot 2+\ldots+15\cdot 6}{10}\\ &= 0.21\cdot 1+0.14\cdot 2+\ldots+0.15\cdot 6\\ &= 3.33.\end{aligned}$$

Note how the mean is expressed by the relative frequencies. In an 'ideal' sample, the relative frequencies would be the probabilities. In that sample, we would find

$$\begin{aligned}\mu &= \frac{1}{6}\cdot 1+\frac{1}{6}\cdot 2+\ldots+\frac{1}{6}\cdot 6\\ &= \frac{1}{6}\cdot\frac{7\cdot 6}{2}=3.5.\end{aligned}$$

This idealised mean is called *expectation value*. It is a theoretical quantity that describes a probability distribution.

> **Rule:** If a random variable X takes different values $x_1,\ldots,x_n$ with probabilities $p_1,\ldots,p_n$ respectively, then its expectation value is
>
> $$\mu = p_1x_1+p_2x_2+\ldots+p_nx_n=\sum_{i=1}^{n}p_ix_i.$$

A measure of the spread of a distribution, we can use the *variance*

$$\begin{aligned}\sigma^2 &= \sum_{i=1}^{n}p_i(x_i-\mu)^2\\ &= p_1(x_1-\mu)^2+\ldots+p_n(x_n-\mu)^2\end{aligned}$$

or the *standard deviation* $\sigma=\sqrt{\sigma^2}$, i.e.

$$\sigma=\sqrt{\sum_{i=1}^{n}p_i(x_i-\mu)^2}.$$

In the example, the variance of the number on a die is

$$\sigma^2 = \frac{1}{6}\left((1-3.5)^2 + (2-3.5)^2 + \ldots + (6-3.5)^2\right) = \frac{35}{12} \approx 2.92.$$

The standard deviation is

$$\sigma \approx \sqrt{2.29} = 1.71$$

3.3 The binomial distribution

Example 3.7. *We consider a lottery in which we have a chance of winning of $\frac{1}{3}$ in every game. We assume that consecutive games are independent.*

1. *I play five games. What is the probability that I win exactly twice?*

2. *What is the probability that I win at most twice in the same game as in the first question.*

3. *The probability to win at least three times is*

3.3.1 General formulas

Consider an experiment[2] that succeeds with probability p and repeat the experiment n times independently. Let X be the random variable counting the successes. Then, the probability $P(X = k)$, i.e. of k successes, is given by

Using the rules from Section 3.0.1, we get

$$P(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$$

and

$$\begin{aligned} P(x \geq k) &= \sum_{i=k}^{n} \binom{n}{k} p^i (1-p)^{n-i} \\ &= 1 - P(X \leq k-1) \end{aligned}$$

The expectation value of X is np and the variance is $np(1-p)$.

[2]Such experiments are called Bernoulli experiment.

3.3.2 A second example

Example 3.8. *The probability that a car travelling along a certain road will have a tyre burst is* $\frac{1}{50}$*. Find the probability that amongst 15 cars that*

1. *exactly one has a burst.*

2. *two or more have a burst.*

3.3.3 The Poisson distribution

Consider an experiment for which the probability p of a success is very small, but the number of n of repetitions is very large, such that the expected number of successes $\lambda = np$ is fixed.

Then to a good approximation, we have

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!},$$

where X counts again the number of successes. This is called the Poisson distribution.
The random variable can take values $k \in \mathbb{N}_0 = \{1, 2, \ldots\}$.
The expected value of X is λ and the variance is λ.
We can quickly check the normalisation:

$$\sum_{k=0}^{+\infty} P(X = k) = \sum_{k=0}^{+\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\underbrace{\sum_{k=0}^{+\infty}\frac{\lambda^k}{k!}}_{=e^{\lambda}} = 1.$$

3.3.4 Two examples

Example 3.9. *On average, 120 cars pass a checkpoint per hour. Let N be the number of cars that pass in a 5-minutes interval.*

1. *What is the probability that exactly 4 cars pass?*

2. *What is the probability that exactly 10 cars pass?*

3. *What is the probability that at least 4 cars pass?*

Example 3.10. *Rocks on the surface of the moon are scattered at random but on average there are 0.3 rocks per M^2 on the moon.*

1. *An exploring vehicle covers an are of $8m^2$. What is the probability that it finds two or more rocks?*

2. *What area should be explored if there is to be a probability of 0.8 of finding 1 or more rocks.*

3.3.5 Continuous probability distributions

In contrast to the binomial and Poisson distribution, we now allow the random variable to take real values and not just integers. That is what the continuous is referring to. Binomial and Poisson are called discrete probability distributions.

Example 3.11. *Spin a wheel of fortune. It can stop at any angle from 0 to 2π. We assume that it stops 'everywhere with same probability'.*

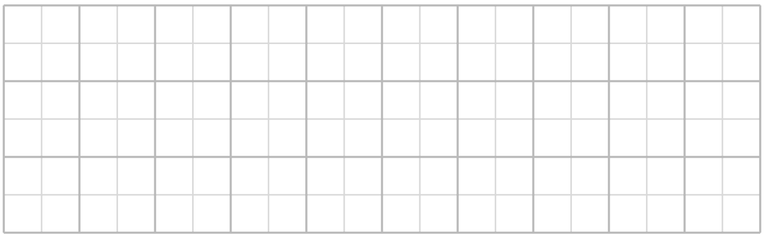
The probability that it stops at exactly π (or at any other point) is zero. We must ask about intervals. The probability that it stops between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ is $\frac{1}{2}$:

$$P(\tfrac{\pi}{2} < \varphi < \tfrac{3\pi}{2}) = P(\tfrac{\pi}{2} \leq \varphi \leq \tfrac{3\pi}{2}) = \frac{1}{2}.$$

The probability that it sops between $\frac{\pi}{2}$ and π is $\frac{1}{4}$.

We describe the distribution by a probability density function, in Example 3.11 and in general. Here, the density would be

This is



Probability density functions have in general the following properties:

1. $f(\varphi) \geq 0$ for all $\varphi \in \mathbb{R}$,
2. $\int_a^b f(\varphi)dx$ is the probability that the wheel stops between a and b. Note: $f(\varphi)d\varphi$ is the probability to obtain a result between φ and $\varphi + d\varphi$.
3. $\int\limits_{-\infty}^{+\infty} f(\varphi)d\varphi = 1$

Mean and variance of a continuous distribution with a pdf (probability density function) f are obtained by

$$\mu = \int\limits_{-\infty}^{\infty} xf(x)dx$$

and

$$\sigma^2 = \int\limits_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

These formulas are similar to the ones for discrete distributions. See Section 3.2.

3.3.6 Examples

Example 3.12. *A random variable X has probability density function*

$$f(x) = \begin{cases} \frac{N}{x^2} & : \quad 1 \leq x \leq 5 \\ 0 & : \quad \textit{otherwise} \end{cases}$$

1. *Find the normalisation constant N.*

2. *Find the probability that X is between 2 and 3.*

3. *The probability that X is larger than 4.*

3.4 The normal (Gaussian) distribution

A random variable X follows a normal distribution with mean μ and standard deviation σ (i.e $X \sim N(\mu, \sigma^2)$) if it has the pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Exercise 3.1. *A company manufactures light bulbs with nominal (mean) power of* $\mu = 10$*W. The actual powers, denoted by* X*, follow a normal distribution with standard deviation* $\sigma = 0.5$*W.*

1. *What is the probability that a light bulb has power between* 10*W and* 11*W?*

2. *What is the probability that power is between 11W and 11.5W?*

3. *What is the probability that the power is between 9.5W and 11W?*

4. *What is the probability that the power is less than 9W.?*

Appendix

A Some Prerequisites

A.1 Integration by substitution

Integration by substitution can be seen as a reversion of the chain rule. If we have an integral of the form

$$\int g'(f(x))f'(x)dx,$$

we can set $y = f(x)$ and obtain $dy = f'(x)dx$. Thus, we get

$$\int g'(f(x))f'(x)dx = \int g'(y)dy.$$

If the last integral is easy, then we integrate and substitute back and finally obtain

$$\int g'(f(x))f'(x)dx = g(f(x)) + C.$$

An example is

$$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int 2x\sqrt{x^2+1}dx.$$

Setting $y = x^2 + 1$, we get $dy = 2xdx$ which gives

$$\int 2x\sqrt{x^2+1}dx = \int \sqrt{y}dy = \frac{2}{3}y^{\frac{3}{2}}.$$

Thus,

$$\int x\sqrt{x^2+1} = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C.$$