Review: Functions

1 Some notation

Sets are collections of elements described by some property P. The standard notation for sets is

$$A = \{x : x \text{ has property } P\},$$

where one reads: A consists of all x such that x has property P. Some sets as the real numbers or the whole numbers have special symbols. See Table 1.

Symbol	Description
$\mathbb{R}$	real numbers
$\mathbb{Z}$	whole numbers
$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	natural numbers containing 0
$\mathbb{Q}$	rational numbers
$\mathbb{C}$	complex numbers

Table 1: Notation of certain sets.

The most important sets of this lecture are sub-sets of the real numbers, called intervals. An interval is a set of numbers characterized by their left and right "boundary". For example

$$[a,b] = \{x \in \mathbb{R} : \quad a \le x \le b\}$$

which we read as *closed interval* a, b. Closed means that it contains a and b. An open interval does not contain the boundary points, i.e.

$$(a,b) = \{x \in \mathbb{R} : \quad a < x < b\}.$$

One can also consider the half-open cases

$$[a,b) = \{x \in \mathbb{R} : \quad a \le x < b\}$$

and

$$(a,b] = \{x \in \mathbb{R} : \quad a < x \le b\}.$$

We also denote $\mathbb{R} = (-\infty, +\infty)$. All numbers smaller than a would be denoted by $(-\infty, a)$, all numbers smaller or equal to a by $(-\infty, a]$. Similarly, one defines the sets of all numbers larger that or larger or equal to a given number. If we have the situation that we describe x as having either the property $x \ge a$ or $x \le -a$ for a given $a \ge 0$, then we can write

$$\{x \in \mathbb{R} : \quad x \ge a \quad \text{or} \quad x \le -a\}$$

which is the same as

$$x \in (-\infty, -a] \cup [a, +\infty).$$

We further introduce the following notation: let $a \in \mathbb{R}$, then

$$\begin{aligned}
\mathbb{R}_{<a} &= (-\infty, a) = \{x \in \mathbb{R} : x < a\}, \\
\mathbb{R}_{\le a} &= (-\infty, a] = \{x \in \mathbb{R} : x \le a\}, \\
\mathbb{R}_{>a} &= (a, +\infty) = \{x \in \mathbb{R} : x > a\}, \\
\mathbb{R}_{\ge a} &= [a, +\infty) = \{x \in \mathbb{R} : x \ge a\}.
\end{aligned}$$

Definition 1.1 (Intersection/Union/Difference). *We denote by $A \cap B$ the intersection of A and B which means that $A \cap B$ contains elements that are in A as well as in B. By $A \cup B$, we denote the union of the two sets A and B which means that $A \cup B$ contains elements that are either in A or in B. With $A \setminus B$, we denote finally the difference of A and B that means that $A \setminus B$ contains all elements in A that are not in B.*

Exercise 1.1. *Write down examples. Do you understand the notation? Write the set of all* x *with* $-1 < x \leq 3$, $9 \leq x \leq 15$, *and* $x > 15$ and $x < -15$ *as interval or union of intervals.*

2 What is a function?

Definition 2.1 (Functions)**.** *A functions is a mathematical relationship consisting of a rule linking elements from two sets such that each element from the first set (the domain) links to one and only one element from the second set (the image set or range).*

Remark 2.1. *The vertical line test allows to decide whether a graph represents a function: the graph represents a function if any line drawn parallel to the y-axis cuts the graph in only one point.*

Remark 2.2. *The range or image of a function is, by the definition above, the collection of all values $f(x)$ when x ranges through the domain of f. In a graph, where you draw $(x, y = f(x))$ in a xy-grid, the domain is the collection of all the x values and the range is the collection of all the y values.*

Examples of typical functions are

$f(x) =$	Domain/image	Special properties	HELM ref.
e^x	Domain: $\mathbb{R}$, image: $\mathbb{R}_{>0}$	Strictly increasing, convex	HELM 6
$\ln(x)$	Domain $\mathbb{R}_{>0}$, image: $\mathbb{R}$	Strictly increasing, concave	HELM 6
$\sum_{k=0}^{n} a_k x^k$	Domain: $\mathbb{R}$, image: $\mathbb{R}$ if n is odd.	Properties depend on the degree. Work out examples!	HELM 3
$\sin(x)$	Domain: $\mathbb{R}$, image: $[-1, 1]$	Periodic with period 2π.	HELM 4
$\cos(x)$	Domain: $\mathbb{R}$, image: $[-1, 1]$	Periodic with period 2π.	HELM 4
$\tan(x)$	Domain: $\bigcup_{k\in\mathbb{Z}} \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, image: $\mathbb{R}$	Periodic with period π.	HELM 4

Table 2: Some typical functions.

Functions can be classified by different means. We will use the following properties

Definition 2.2 (Properties of Functions)**.** *Let f be a real-valued function.*

- *f is called strictly monotonically increasing if $f(x) > f(y)$ for all $x > y$ and strictly monotonically decreasing if $f(x) < f(y)$ for all $x > y$. Example: The function $f(x) = e^x$ is strictly increasing and $f(x) = e^{-x}$ is strictly decreasing.*
- *f is called even if $f(x) = f(-x)$ and odd if $f(-x) = -f(x)$. Example: The function $f(x) = \sin(x)$ is odd and $f(x) = \cos(x)$ is even.*
- *f is called continuous if you can draw it without taking the pen from the paper. In particular, continuous functions can* not *have jumps as for example in the function*

$$f(x) = \begin{cases} 1, x \leq 0 \\ 3, x > 0 \end{cases}$$

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2.1 Operations on functions

We can add, subtract, multiply, and divide functions by pointwise definition.Another important operation is the composition: let $f : A \to B$ and $g : B \to C$ be functions. Then, $g \circ f(x) = g(f(x))$ is a function from A to C. Unless C is contained in A, the function $f \circ g$ is not defined.

Example 2.1. *Let $f(x) = e^x$ and $g(x) = x^2 + 1$. Both functions are defined on $\mathbb{R}$. We can consider*

$$f \circ g(x) = f(g(x)) = e^{x^2+1},$$
$$g \circ f(x) = g(f(x)) = e^{2x} + 1.$$

That also shows that, in general, $f \circ g$ is not equal to $g \circ f$ should both be definable.

Remark 2.3. *There are resources that use $fg(x)$ instead of $f \circ g(x)$ as symbol for $f(g(x))$. Since we have introduced $fg(x)$ as a symbol for the product of f and g (the most natural interpretation), we will not use it for $f(g(x))$ in this class and you should in general discard it. Otherwise you will confuse yourself and others.*

3 What is the inverse of a function?

Definition 3.1 (One-one function)**.** *A function f is called one-one if every element of the domain is linked to a unique element of the image, i.e. $f(x) = f(y)$ implies $x = y$.*

Remark 3.1. *The horizontal line test allows to decide whether a graph represents a one-one function. If all lines drawn parallel to the x-axis cut the graph at most once, the represented function is one-one.*

Definition 3.2 (Inverse function)**.** *Let f be a one-one function. Then, there exists a function g, called the inverse function to f, such that $g(f(x)) = f(g(x)) = x$. We denote the function g by f^{-1} (not to be confused with $1/f$). The domain of the function f^{-1} is the image of f and the image of f^{-1} is the domain of f.*

Further reading on inverse functions, may be found in HELM 2.

Function f	Inverse function f^{-1}	HELM ref.
$e^x : \mathbb{R} \to R_{>0}$	$\ln(x) : \mathbb{R}_{>0} \to \mathbb{R}$	HELM 3
$\sin(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$	$\arcsin(x) : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	HELM 4
$\cos(x) : [0,\pi] \to [-1,1]$	$\arccos(x) : [-1,1] \to [0,\pi]$	HELM 4
$\tan(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to (-\infty, +\infty)$	$\arctan(x) : (-\infty, +\infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	HELM 4

Table 3: Some typical functions with inverses.

3.1 How to compute the inverse?

Let us explain the computation of an inverse of a function by an example. Let f be given by

$$f(x) = 3x + 15.$$

First, one needs to make sure that the functions is one-one. The horizontal line test allows us to conclude, that f is one-one on its domain. We interchange x and y and solve the resulting equation for y:

$$y = f(x) = 3x + 15 \quad \Rightarrow \quad x = 3y + 15$$

which leads to

$$y = f^{-1}(x) = \frac{x}{3} - 15.$$

What if the horizontal line test fails, e.g. for $\sin(x)$? It may still be possible to invert the function on a subset of the domain. The strategy is to look for monotonicity intervals, i.e. parts of the domain n which the function is either strictly monotonically increasing or strictly monotonically decreasing.

Example 3.1. *As an exercise, let me exemplify how a complete solution of a problem from the problem sheet looks like: Consider Problem 2(3) on the Problem sheet* Functions*. The function $f(x) = \cos(x)$ is given with the domain $(-\pi, \pi)$, i.e. we are allowed to plug all x into f that satisfy $-\pi < x < \pi$. That implies that values $f(x)$ satisfy $-1 < f(x) < 1$.*[1] *Therefore, the image of f is the set $(-1,1)$. The cosine is not invertible on the entire domain since it is symmetric, i.e. $\cos(x) = \cos(-x)$ for all $x \in (-\pi, \pi)$* [2]*. However, the function is strictly increasing on $(-\pi, 0]$ and strictly decreasing on $[0, \pi)$. Hence, we can invert it, for example, on those intervals. Thus, we consider*

$$y = \cos(x), \quad x \in [0, \pi).$$

[1]Why? Look at the unit circle and what happens with $\cos(x)$ as the "angle" varies between $-\pi$ and π. (Draw pictures!)
[2]Why not? Well, x and $-x$ get the same value assigned. Thus, the horizontal line test must fail.

To compute the inverse, we exchange the variables x and y and solve for y:

$$x = \cos(y) \quad \Rightarrow \quad y = f^{-1}(x) = \arccos(x),$$

where $-1 < x < 1$ [3] and $0 \leq y < \pi$.

[3] Keep in mind the swap of x and y.