Tangents and tangent planes

The equation for a tangent on the graph of a given function f at a point x_0 is give by

$$\begin{aligned} T_{f,x_0}(x) &= f(x_0) + f'(x_0)(x - x_0), && (1) \\ &= f'(x_0)x + f(x_0) - f'(x_0)x_0. && (2) \end{aligned}$$

From (1), we can read off two key properties:

- $T_{f,x_0}(x_0) = f(x_0)$, which simply means that the tangent takes the same function value as f at x_0.
- The function $T'_{f,x_0}(x_0) = f'(x_0)$. This means that the function T_{f,x_0} has the same slope at x_0 as the function f.

Additionally, from (2), we can read off that T_{f,x_0} cuts the y-axis at $f(x_0) - f'(x_0)x_0$.

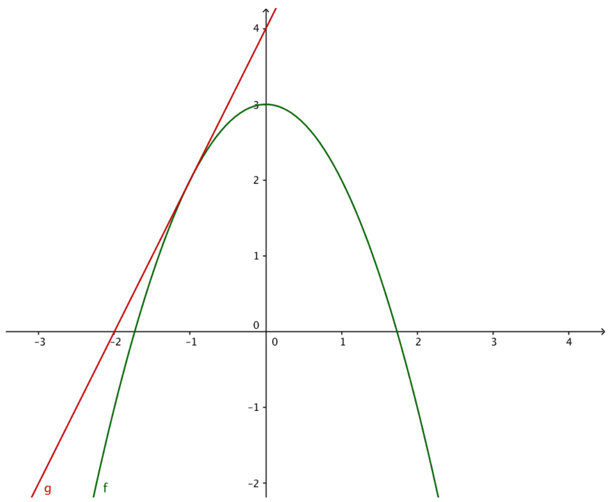

Figure 1: The red line is tangent to the green curve at $x_0 = -1$.

The two properties above do in fact characterize the tangent on a graph of a function at a point x_0. That means that if we start with a general linear function $T(x) = mx + n$ and want to find m and n such that

- $T(x_0) = f(x_0)$ and
- $T'(x_0) = f'(x_0)$.

From our ansatz, we obtain $T'(x) = m$ which leads to $m = f'(x_0)$ from the second condition. The first condition implies

$$T(x_0) = f(x_0) = mx_0 + n \quad \Rightarrow \quad n = f(x_0) - f'(x_0)x_0,$$

where we used $m = f'(x_0)$. Thus, we get that $T(x) = T_{f,x_0}(x)$ for all x. The material on tangents can also be found in HELM 12.1.

Remark 0.1
Remember that the value $f'(x_0)$ is also the tangent of the angle θ that the tangent on f at x_0 encloses with the x-axis:

$$\tan(\theta) = f'(x_0).$$

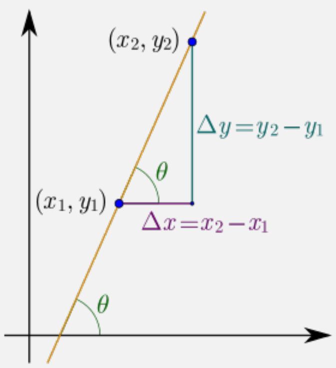

Figure 2: The picture was created by Maschen under CC BY-SA 3.0. It shows the above described propery.

Now we want to do the same for a given $f(x, y)$ which depend not only on one but on two variables. Now we do not find a tangent line but a tangent plane. Again, we would like the plane to take the value $f(x_0, y_0)$ at a given point (x_0, y_0).

First, we make an ansatz for a general plane:

$$P(x, y) = ax + by + d. \tag{3}$$

Secondly, we ask the following two properties:

- $P(x_0, y_0) = f(x_0, y_0)$ and
- $\nabla P(x_0, y_0) = \nabla f(x_0, y_0) = \begin{bmatrix} \partial_x f(x_0, y_0) \\ \partial_y f(x_0, y_0) \end{bmatrix}.$

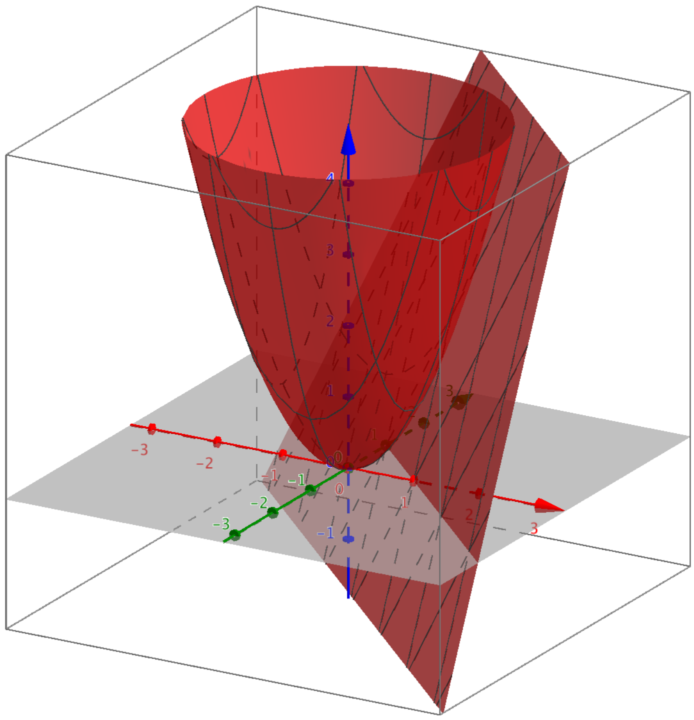

Figure 3: The Tangent plane $P(x, y)$ on the function $f(x, y) = x^2 + y^2$ is at $(1, 1)$.

The first condition gives us with (3) that

$$P(x_0, y_0) = f(x_0, y_0) = ax_0 + by_0 + d \quad \Rightarrow \quad d = f(x_0, y_0) - ax_0 - by_0.$$

From (3), we also get

$$\nabla E(x, y) = \begin{bmatrix} \partial_x E(x, y) \\ \partial_y E(x, y) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4}$$

Plugging (x_0, y_0) into (4), we get, together with the second condition that

$$a = \partial_x f(x_0, y_0), \quad b = \partial_y f(x_0, y_0).$$

Putting everything together, we obtain

$$\begin{aligned} P(x, y) &= ax + by_0 + d, \\ &= \underbrace{\partial_x f(x_0, y_0)}_{=a} x + \underbrace{\partial_y f(x_0, y_0)}_{=b} y \\ &\quad + \underbrace{f(x_0, y_0) - \partial_x f(x_0, y_0)x_0 - \partial_y f(x_0, y_0)y_0}_{=d}, \\ &= f(x_0, y_0) + \partial_x f(x_0, y_0)(x - x_0) + \partial_y f(x_0, y_0)(y - y_0). \end{aligned}$$

Using vector notation, one can write the last line as

$$P(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$

The scalar product[1] $\cdot$ of two vectors $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is given by $x \cdot y = x_1 y_2 + x_2 y_2$.

Thus, we have the following theorem

Theorem 0.1 (Tangents and tangent planes)
In this theorem, we describe the formulae for the tangent on a graph of a function of one variable and the tangent plane on a graph of a function of two variables.

- *Let f be a function of one variable and let x_0 be a point in the domain of f. Then, the tangent on the graph of f at the point x_0 is given by*

$$T(x) = f(x_0) + f'(x_0)(x - x_0).$$

- *Let f be a function of two variables and let (x_0,y_0) be a point in the domain of f. Then, the tangent plane on the graph of f at the point (x_0,y_0) is given by*

$$P(x,y) = f(x_0,y_0) + \partial_x f(x_0,y_0)(x - x_0) + \partial_y f(x_0,y_0)(y - y_0).$$

The use of tangents and tangent planes os that one can replace the function f by T or P if x, respectively (x,y) is sufficiently close to x_0 respectively (x_0,y_0). This is also called linear approximation.

[1] In school sometimes called dot product.